

A Nonlinear Saturating Amplifiers Circuit for Piezoelectric Vibration Control of Flexible Mechanical System

Kevin Dekemele, Christophe Giraud-Audine, Olivier Thomas

Abstract. This work investigates the vibration control properties of a saturating shunt circuit for piezoelectric vibration control. The circuit is realized as a saturating operational amplifier that takes the induced piezoelectric voltage as input. Furthermore, an inductor and resistor are placed in series. Using complexification-averaging, an analytical form of the frequency response is derived by approximating the saturation by an arctangent. It is found that the vibration damping properties are similar to that of the piezo-electric nonlinear energy sink (NES) shunt circuit that consists of a cubic multiplication of the piezoelectric voltage. These properties are 1) Saturation of the host structure amplitude 2) an efficient quasi-periodic vibration regime, and 3) the appearance of an isolated response curve (IRC). The difference is that for the saturating circuit, the isolated response appears to the right of the host structures' resonance, while for the NES circuit, it results in an IRC to the left of this resonance. The analytical frequency response is compared to Runge-Kutta simulation for arctangent and non-smooth saturation, showing good agreement.

Key words: Piezoelectric Vibration Control, Shunt Circuit, Nonlinear circuit

1 INTRODUCTION

In vibration control, the nonlinear energy sinks (NES) devices are mechanical vibration absorbers that have a wider operational frequency bandwidth compared to linear counterparts as Tuned-Mass-Dampers [1]. This increased performance is attributed to the nonlinear spring, typically polynomial hardening. This hardening spring can be created by many mechanisms and is one of the main reasons this type of spring is common in investigations. In particular, the beneficial effects of the NES with a host system under harmonic excitation are a saturation of host system amplitude regardless of force magnitude, and an efficient quasi-periodic vibration regime called strongly modulated response (SMR). However, if the applied force is too large, an isolated resonance curve (IRC) appears to the left of the host system's natural frequency,

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detrimental to the NES performance. In piezoelectrical vibration control, the common linear resistor-inductor (RL) shunt behaves similarly as a tuned-mass-damper [2] and similarly suffers from a limited effective frequency bandwidth. Although nonlinear shunt circuits have been investigated, they did not show the beneficial amplitude saturation and SMR response regime [3, 4, 5, 6, 7]. Recently, a NES shunt circuit for piezoelectric vibration control was investigated and experimentally verified. It has the same behavior as a mechanical NES [8], that is, saturation of host system vibration amplitude, SMR, and the appearance of an IRC. The hardening characteristic was achieved by a cubic voltage law realized electronically by two analog multiplier components. The saturation effect of the host structure's amplitude was achieved near the open circuit resonance, but an IRC also appears to the left of this resonance. Although that work has shown an electronic NES is possible, in electronics, polynomial nonlinearities are not so common and was artificially created with the multiplier circuit. In this paper, we will investigate a saturating amplifier shunt circuit, a nonlinearity common in electronics yet challenging to realize as a mechanical mechanism. Some numerical investigations do exist for mechanical saturating stiffness, yet an experimental verification is still missing [9, 10]. We will derive the frequency response with complexification averaging and show that a saturation effect near the resonance also exist, this time near the short-circuited resonance. Around resonance, SMR also occurs. If the force magnitude is increased, an IRC appears, but unlike for the polynomial hardening circuit, this time to the right on the resonance instead of the left. For applications where excitation frequency ramps up, this property is especially interesting. The analytical frequency response is compared with the results from a multi-harmonic harmonic balancing toolbox and with the numerical results of a Runge-Kutta simulation of the full equations of motion.

2 THEORETICAL DESCRIPTION AND TUNING

2.1 EQUATIONS OF MOTION

The studied system is a linear elastic structure with a bonded piezoelectric patch, schematically shown in Figure 1a. The electrodes of the piezoelectric patch are shunted by a nonlinear circuit, consisting of a resistor, inductor and a nonlinear voltage source. The nonlinear voltage source under consideration in this work is a saturation nonlinearity, as shown in Figure 1b, which will be approximated by an arctangent in order to find an analytical solution.

The equations of motion (EOM) describing the dynamics of the coupled elastic structure, piezoelectric patch and shunt circuit is:

$$\begin{aligned} \mathbf{M}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{K}\mathbf{q} + \Theta V &= \mathbf{F} \cos(\omega t) \\ C_p V - Q + \Theta^T \mathbf{q} &= 0 \\ V + L\ddot{Q} + R\dot{Q} + V_{nl} &= 0 \end{aligned} \tag{1}$$

where $\mathbf{q} \in \mathbb{R}^{n \times 1}$ is the modal displacement vector, $\mathbf{M}$, $\mathbf{C}$, $\mathbf{K}$ are the modal mass, modal damping and modal stiffness matrices, all $\in \mathbb{R}^{n \times n}$ and $\mathbf{F} \in \mathbb{R}^{n \times 1}$ is the modal force vector. On the piezoelectrical side, $\Theta \in \mathbb{R}^{n \times 1}$ is the modal electromechanical coupling vector, V is the voltage over the piezo's electrodes, Q is the charge at the electrodes, C_p is the piezoelectrical

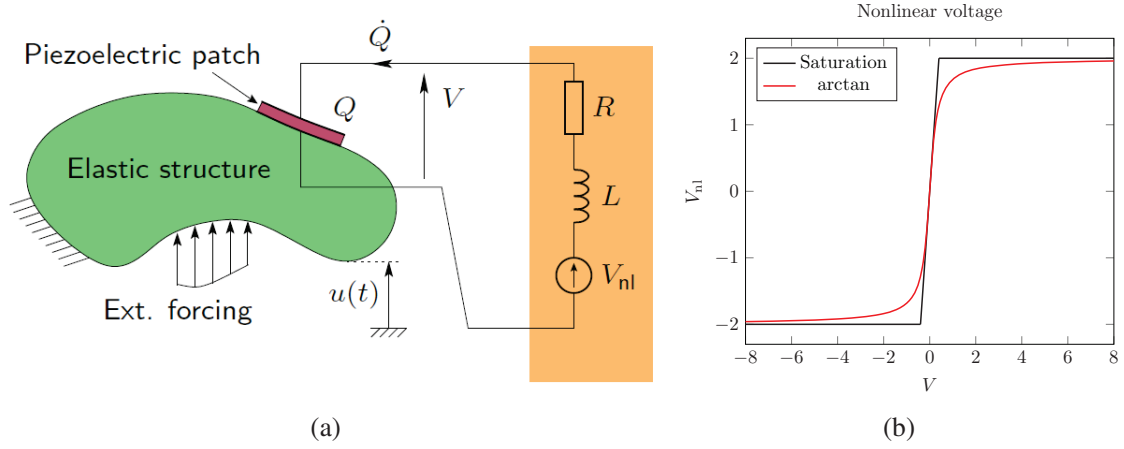


Figure 1: The system under consideration (a) and the studied nonlinearity (b).

capacitance and R and L are the shunt's resistance and inductance. The nonlinear voltage source in the shunt is a saturating voltage, that can be approximated by a arctan:

$$V_{nl} = \begin{cases} \beta_g V, & -\frac{V_{sat}}{\beta_g} < V < \frac{V_{sat}}{\beta_g} \\ V_{sat} & V > \frac{V_{sat}}{\beta_g} \\ -V_{sat} & V < -\frac{V_{sat}}{\beta_g} \end{cases} \approx V_{sat} \frac{2}{\pi} \arctan\left(\frac{\pi \beta_g}{2 V_{sat}} V\right) \quad (2)$$

The approximation of the saturation component by the smooth arctan voltage allows for an analytical derivation of the frequency response later on.

The dynamical behavior around a single linear mode is studied, which reduces (1) to the following equations of motion:

$$\begin{aligned} \ddot{q}_i + 2\zeta_i \omega_i \dot{q}_i + \omega_i^2 q_i + \frac{\theta_i}{m_i C_p} Q &= \frac{F_i}{m_i} \cos(\omega t) \\ \ddot{Q} + 2\zeta_e \omega_e \dot{Q} + \omega_e^2 Q + \frac{\theta_i}{L C_p} q_i + \frac{V_{nl}}{L} &= 0 \end{aligned} \quad (3)$$

where m_i is the modal mass of mode i , ω_i is the short-circuit eigenfrequency of mode i , ζ_i is the modal damping of mode i , θ_i is the modal electromechanical coupling of mode i , ω_e is the electrical eigenfrequency, and ζ_e the electrical damping coefficient. The last two are defined as:

$$\omega_e = \sqrt{\frac{1}{L C_p}} \quad \zeta_e = \frac{R}{2} \sqrt{\frac{C_p}{L}} \quad (4)$$

2.2 DIMENSIONLESS EQUATIONS

By defining the following dimensionless modal displacement, charge voltage and time:

$$\bar{q}_i = \frac{\sqrt{m_i}\omega_i}{\sqrt{C_p}} \frac{\pi\beta_g}{2V_{\text{sat}}} q_i, \quad \bar{Q} = \frac{\sqrt{L}\omega_i}{\sqrt{C_p}} \frac{\pi\beta_g}{2V_{\text{sat}}} Q, \quad \bar{V} = \frac{\pi\beta_g}{2V_{\text{sat}}} V, \quad \tau = \omega_i t, \quad (5)$$

Eq. (3) becomes:

$$\begin{aligned} \bar{q}_i'' + 2\zeta_i \bar{q}_i' + \bar{q}_i + k_i r_i \bar{Q} &= \bar{F}_i \cos(\Omega\tau) \\ \bar{Q}'' + 2\zeta_e r_i \bar{Q}' + r_i^2 \bar{Q} + k_i r_i \bar{q}_i + \beta_g r_i \arctan(\bar{V}) &= 0 \end{aligned} \quad (6)$$

where

$$\omega' = \frac{d\omega}{d\tau} \quad r_i = \frac{\omega_e}{\omega_i} \quad \bar{F}_i = \frac{\pi\beta_g}{2V_{\text{sat}}} \frac{F_i}{\omega_i \sqrt{m_i} \sqrt{C_p}} \quad \Omega = \frac{\omega}{\omega_i} \quad k_i^2 = \frac{\theta_i^2}{\omega_i^2 C_p m_i} \quad (7)$$

In the next step, only the voltage is kept as electrical coordinate, where $\bar{Q}$ is eliminated by $\bar{V} = (k_i \bar{q}_i + r_i \bar{Q})$:

$$\begin{aligned} \bar{q}_i'' + 2\zeta_i \bar{q}_i' + (1 - k_i^2) \bar{q}_i + k_i \bar{V} &= \bar{F}_i \cos(\Omega\tau) \\ \bar{V}'' - k_i \bar{q}_i'' + 2\zeta_e r_i (\bar{V}' - k_i \bar{q}_i') + r_i^2 \bar{V} + \beta_g r_i^2 \arctan(\bar{V}) &= 0 \end{aligned} \quad (8)$$

Opting for $\bar{V}$ as electrical variable allows for a closed form of the frequency response that is derived in the next section.

3 FREQUENCY RESPONSE

3.1 COMPLEXIFICATION AVERAGING (CxA)

The modal coordinate and piezoelectrical voltage are assumed to be in 1:1 resonance. The complex variables $A \in \mathbb{C}$ and $B \in \mathbb{C}$ of Manevitch are defined [11]:

$$2A(\tau)e^{j\tau} = \bar{q}_i - j\bar{q}_i' \quad 2B(\tau)e^{j\tau} = \bar{V} - j\bar{V}' \quad (9)$$

where $j = \sqrt{-1}$ is the imaginary unit. The original variables $\bar{q}_i$, $\bar{V}$ and their derivatives in functions of A and B are:

$$\begin{aligned} \bar{q}_i &= A(\tau)e^{j\Omega\tau} + A^*(\tau)e^{-j\Omega\tau} & \bar{V} &= B(\tau)e^{j\Omega\tau} + B^*(\tau)e^{-j\Omega\tau} \\ \bar{q}_i' &= j\Omega(A(\tau)e^{j\Omega\tau} - A^*(\tau)e^{-j\Omega\tau}) & \bar{V}' &= j\Omega(B(\tau)e^{j\Omega\tau} - B^*(\tau)e^{-j\Omega\tau}) \end{aligned} \quad (10)$$

The acceleration and second derivative of the voltage in the function of the new variables A and B is found by time derivation of (9) and using (10):

$$\bar{q}_i'' + \Omega^2 \bar{q}_i = j2\Omega A' e^{j\Omega\tau} \quad \bar{V}'' + \Omega^2 \bar{V} = j2\Omega B' e^{j\Omega\tau} \quad (11)$$

Substituting (10) and (11) into (8), and averaging over the first harmonic yields:

$$\begin{aligned}
2jA'\Omega + (1 - k_i^2 - \Omega^2)A + 2j\zeta_i\Omega A + k_iB &= \frac{\bar{F}_i}{2} \\
2jB'\Omega + (r_i^2 - \Omega^2)B - 2jA'\Omega k_i + \Omega^2 A k_i + 2\zeta_e r_i(j\Omega B - j\Omega A k_i) & \\
+ \beta_g r_i^2 B \frac{\sqrt{4BB^* + 1} - 1}{2BB^*} &= 0
\end{aligned} \tag{12}$$

where the last term is found by projecting $\beta_g r_i^2 \arctan(B(\tau)e^{j\Omega\tau} + B^*(\tau)e^{-j\Omega\tau})$ on a Fourier series, and only keeping the first harmonic:

$$\begin{aligned}
\frac{1}{2\pi} \int_0^{2\pi} f_V(\bar{V})e^{-i\tau} d\tau &= \frac{\beta_g r_i^2}{2\pi} \int_0^{2\pi} \arctan(B(\tau)e^{j\Omega\tau} + B^*(\tau)e^{-j\Omega\tau}) e^{-i\tau} d\tau \\
&= \beta_g r_i^2 B \frac{\sqrt{4BB^* + 1} - 1}{2BB^*}
\end{aligned} \tag{13}$$

3.2 STEADY-STATE EQUATIONS

The equations are studied in steady state, where $A' = B' = 0$. The complex variables A and B are written in their polar forms $A = \frac{a}{2}e^{j\alpha}$ and $B = \frac{b}{2}e^{j\beta}$, and the equation is split in real and imaginary parts. This procedure yields:

$$\begin{aligned}
(1 - k_i^2 - \Omega^2)a + k_i b \cos(\beta - \alpha) &= \bar{F}_i \cos(-\alpha) \\
2\zeta_i \Omega a + k_i b \sin(\beta - \alpha) &= \bar{F}_i \sin(-\alpha) \\
(r_i^2 - \Omega^2)b + \Omega^2 a k_i \cos(\alpha - \beta) + 2\zeta_e r_i k_i \Omega a \sin(\alpha - \beta) + 2\beta_g r_i^2 \frac{\sqrt{b^2 + 1} - 1}{b^2} &= 0 \\
\Omega^2 a k_i \sin(\alpha - \beta) + 2\zeta_e r_i \Omega b - 2\zeta_e r_i k_i \Omega a \cos(\alpha - \beta) &= 0
\end{aligned} \tag{14}$$

Combining the third and fourth equation in (14) obtains a slow invariant manifold (SIM) that relates a and b :

$$k_i^2 (\Omega^4 + (2\zeta_e r_i \Omega)^2) a^2 = \left((2\zeta_e r_i \Omega)^2 + \left((\Omega^2 - r_i^2) - 2\beta_g r_i^2 \frac{\sqrt{b^2 + 1} - 1}{b^2} \right)^2 \right) b^2 \tag{15}$$

An example of the SIM is shown in Figure 2. The minimum and maximum of the SIM are denoted as $[b_-, a_+]$ and $[b_+, a_-]$. These points are associated with the minimum and maximum amplitudes of a during SMR. The point on the right branch of the SIM where $a = a_+$ is $[b_{++}, a_-]$, where b_{++} is the maximum amplitude of b during SMR. Eliminating a from the third and fourth equation in (14) using the first two, yields a relation between the force $\bar{F}_i$ and b :

$$\begin{aligned}
b^2 \left[(1 - k_i^2 - \Omega^2) \left((r_i^2 - \Omega^2) + 2\beta_g r_i^2 \frac{\sqrt{b^2 + 1} - 1}{b^2} \right) - \Omega^2 k_i^2 - 4\Omega^2 \zeta_i \zeta_e r_i \right]^2 & \\
+ b^2 \left[(1 - \Omega^2) 2\Omega \zeta_e r_i + 2\zeta_i \Omega \left((r_i^2 - \Omega^2) + 2\beta_g r_i^2 \frac{\sqrt{b^2 + 1} - 1}{b^2} \right) \right] & \\
- (\bar{F}_i k_i)^2 ((2\Omega \zeta_e r_i)^2 + \Omega^4) &= 0
\end{aligned} \tag{16}$$

With equations (16) and (15), a frequency response is obtained, with examples on Figure 3 for various forcing amplitudes. To compute the stability of the solution, the eigenvalues of the Jacobian of (12) are computed. The full lines are stable solutions, while the dashed lines are unstable solutions. It can be seen that the host system amplitude a saturates on the line of a_+ , which can be computed from the SIM for each frequency. Once the curve folds around this line, the Neimark-Sacker (NS) bifurcation makes the curve unstable, indicating a quasi-periodic response of the full EOM.

To verify the obtained analytical frequency response, it is compared to the result of a harmonic balancing (HB) continuation toolbox called MANLAB [12, 13]. The result of an HB of 30 harmonics is compared to the analytical result in Figure 4. Although they match quite well, the main difference is two side tongues, which are highlighted in the figure. According to the MANLAB, these are attributed mainly to the 3rd harmonic. Furthermore, a series of bifurcations (B) and NS (N) bifurcation suggests asymmetrical branches, which will have to be explored further in future works.

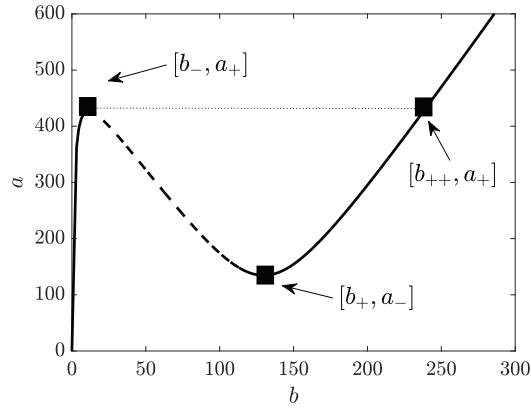


Figure 2: Examples a of a SIM for $k_i = \zeta_e = 0.2$, $r_i = 0.5$, $\zeta_i = 0.005$, $\Omega = 0.98$ and $\beta_g = 200$.

4 COMPARING RK SIMULATIONS AND WITH CXA FR

Here, the full EOM will be simulated with a sine sweep and compared to the analytical frequency response.

For $k_i = \zeta_e = 0.2$, $r_i = 0.5$, $\zeta_i = 0.005$, $\beta_g = 200$, the full dynamical equations are simulated with a force $\bar{F}_i = 20$ and forcing frequency $\Omega = 0.98$. Both the non-smooth saturating and the arctan voltage sources are simulated. The corresponding SIM is shown in Figure 5a, with the envelopes of the RK time simulations extracted from Figures 5b and 5c. This shows a quasi-periodic energy exchange between the mechanical coordinate $\bar{q}$ and the voltage $\bar{V}$. The a_+ and a_- do indeed well predict the maximum amplitude of the mechanical vibration in $\bar{q}$, while it is b_{++} and b_- that predict the maximum and minimum voltage. The RK simulation is now done for a whole frequency range in Figures 6 and 7. For each frequency, the minimum, maximum

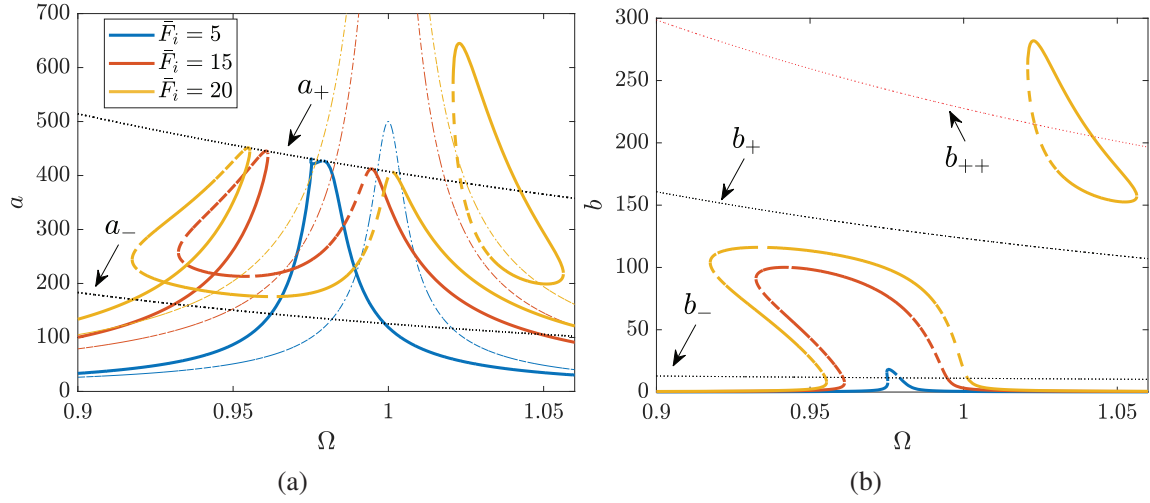


Figure 3: Several frequency response for $k_i = \zeta_e = 0.2$, $r_i = 0.5$, $\zeta_i = 0.005$ and $\beta_g = 200$.

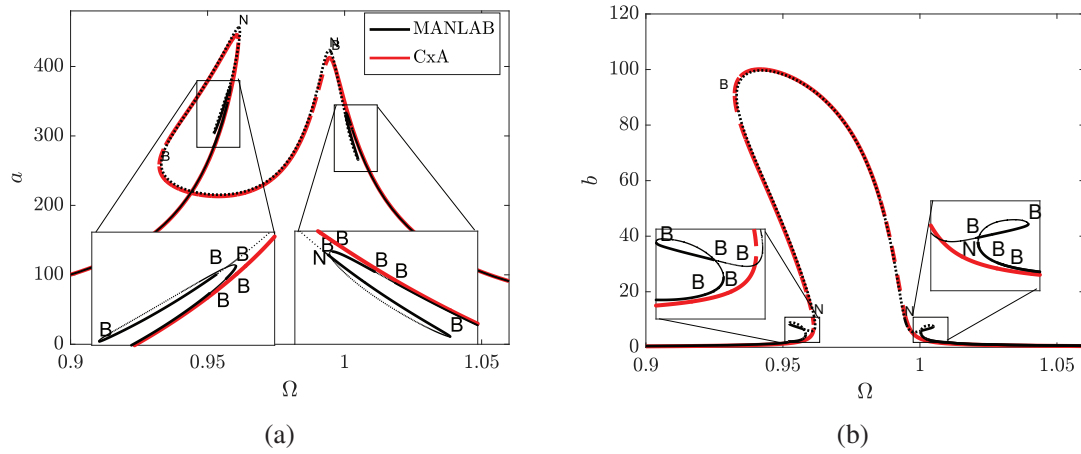


Figure 4: Comparing the frequency response from CxA and MANLAB where $k_i = \zeta_e = 0.2$, $r_i = 0.5$, $\zeta_i = 0.005$, $\beta_g = 200$, $F_i = 15$ and $H = 30$

and RMS values are extracted and plotted on the FR computed from the CxA technique. Good correspondence is seen. Once $\bar{F}_i = 25$, the isola in the FR for a merges with the main branch, and the simulations reflect this.

5 CONCLUSION

This work has investigated the vibration mitigation performance of a saturating voltage source shunt in piezoelectric vibration control. An analytical form was found for the frequency response, and showed good correspondence to the frequency response found with a numerical harmonic balancing continuation toolbox. Similarly like a nonlinear energy sink shunt, the host structure amplitude saturates, there is strongly modulated response and an isolated response curve appears for high forcing. A sweep on the full equations of motion confirm this behavior. As a perspective, a more thorough numerical investigation will be done to further investigated the bifurcations found in MANLAB that suggest assymetric solutions. Furthermore, an experimental verification will be done.

REFERENCES

- [1] H. Ding and L.-Q. Chen, “Designs, analysis, and applications of nonlinear energy sinks,” *Nonlinear Dynamics*, vol. 100, no. 4, pp. 3061–3107, 2020.
- [2] O. Thomas, J. Ducarne, and J.-F. Deü, “Performance of piezoelectric shunts for vibration reduction,” *Smart Materials and Structures*, vol. 21, no. 1, p. 015008, 2011.
- [3] Z. A. Shami, C. Giraud-Audine, and O. Thomas, “A nonlinear piezoelectric shunt absorber with 2: 1 internal resonance: experimental proof of concept,” *Smart Materials and Structures*, vol. 31, no. 3, p. 035006, 2022.
- [4] —, “A nonlinear tunable piezoelectric resonant shunt using a bilinear component: theory and experiment,” *Nonlinear Dynamics*, vol. 111, no. 8, pp. 7105–7136, 2023.
- [5] —, “A nonlinear piezoelectric shunt absorber with a 2:1 internal resonance: Theory,” *Mechanical Systems and Signal Processing*, vol. 170, p. 108768, 2022.
- [6] O. Alfahmi and A. Erturk, “Programmable hardening and softening cubic inductive shunts for piezoelectric structures: Harmonic balance analysis and experiments,” *Journal of Sound and Vibration*, vol. 571, p. 118029, 2024.
- [7] T. M. Silva, M. A. Clementino, C. De Marqui Jr, and A. Erturk, “An experimentally validated piezoelectric nonlinear energy sink for wideband vibration attenuation,” *Journal of Sound and Vibration*, vol. 437, pp. 68–78, 2018.
- [8] K. Dekemele, C. Giraud-Audine, and O. Thomas, “A piezoelectric nonlinear energy sink shunt for vibration damping,” *Mechanical Systems and Signal Processing*, vol. 220, p. 111615, 2024.

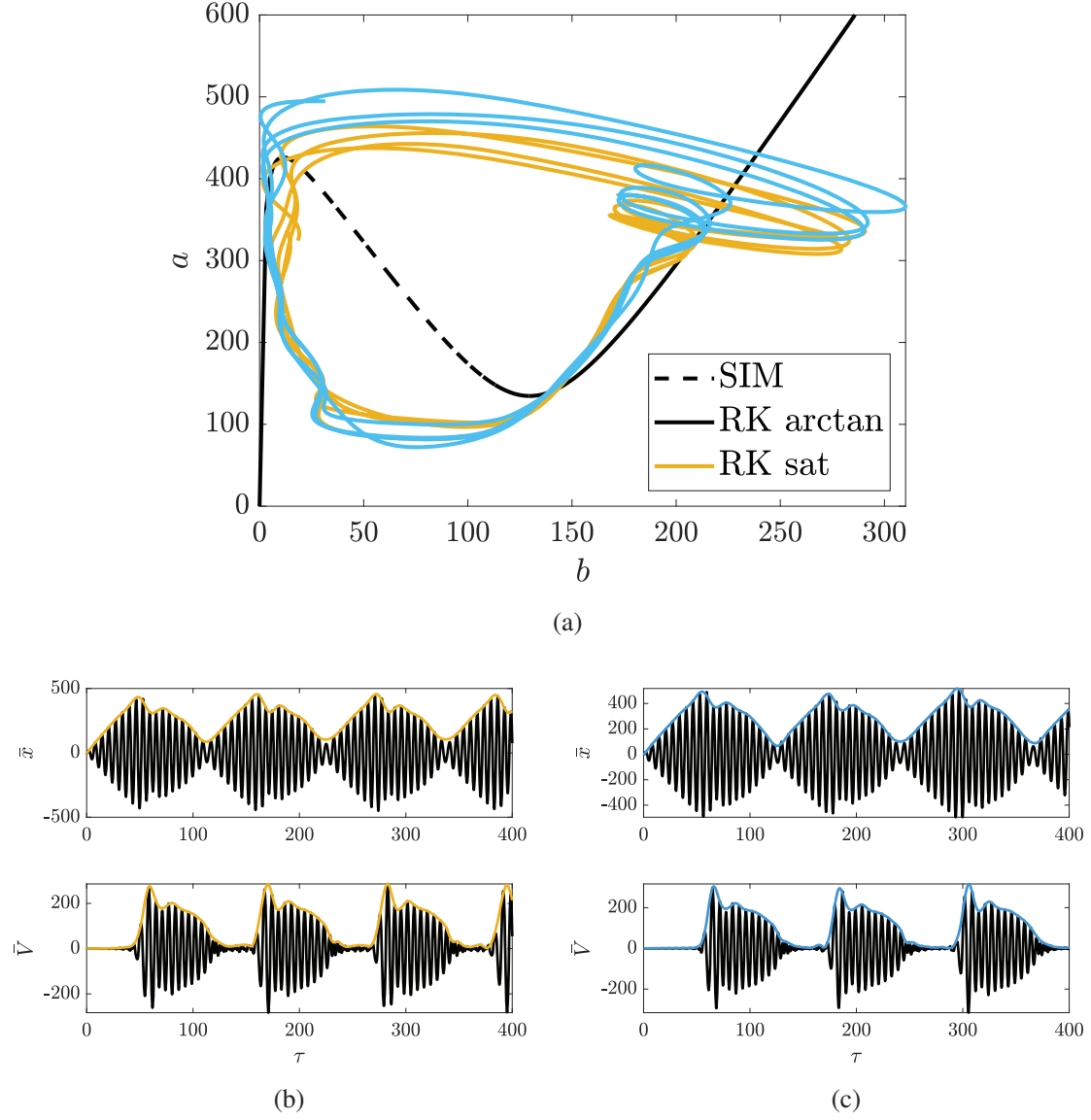


Figure 5: Time simulation for $k_i = \zeta_e = 0.2$, $r_i = 0.5$, $\zeta_i = 0.005$, $\beta_g = 200$, $\bar{F}_i = 20$ and $\Omega = 0.98$.

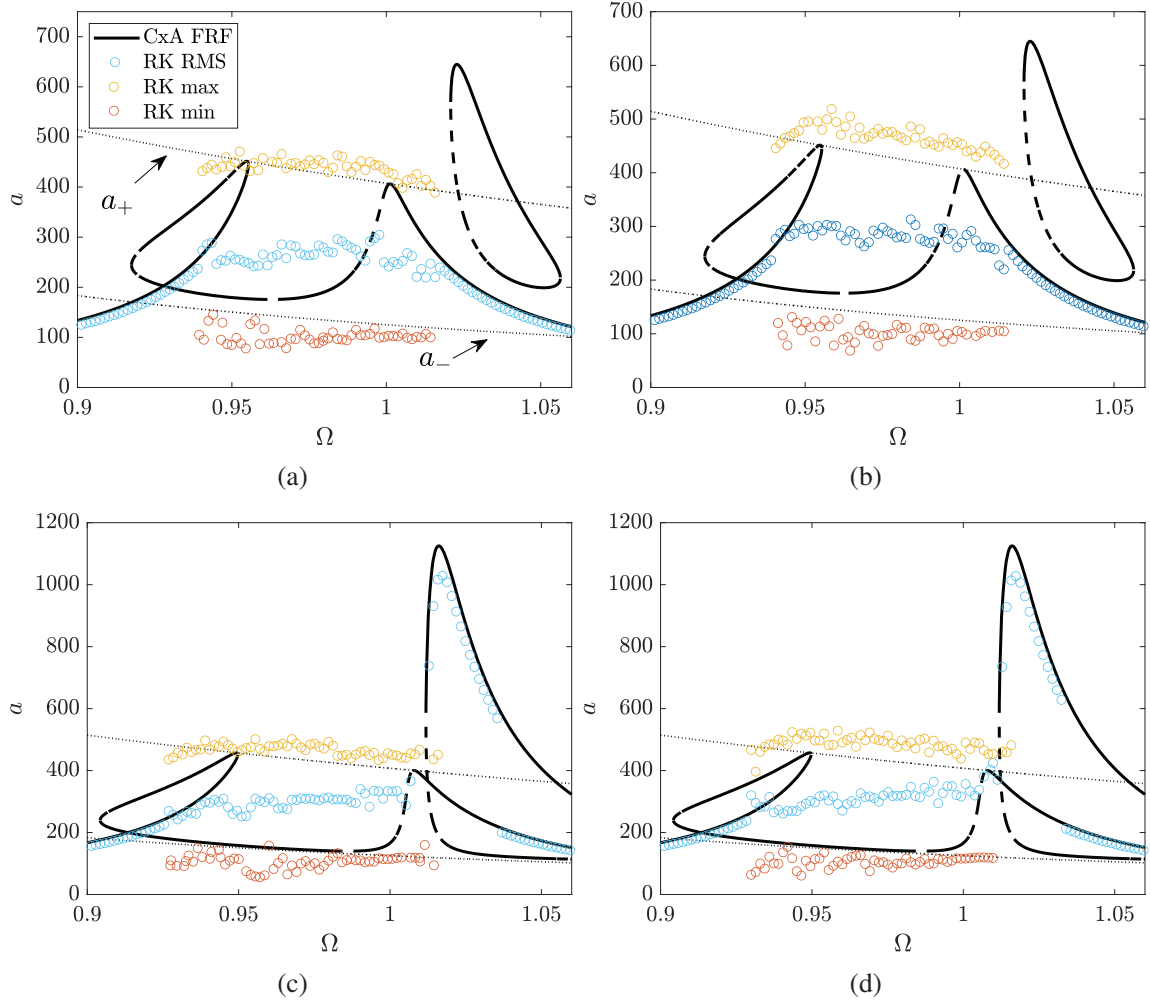


Figure 6: FRFs for a , left side is arctan and right side saturated voltage. (a)(b) $\bar{F}_i = 20$ (c)(d) $\bar{F}_i = 25$.

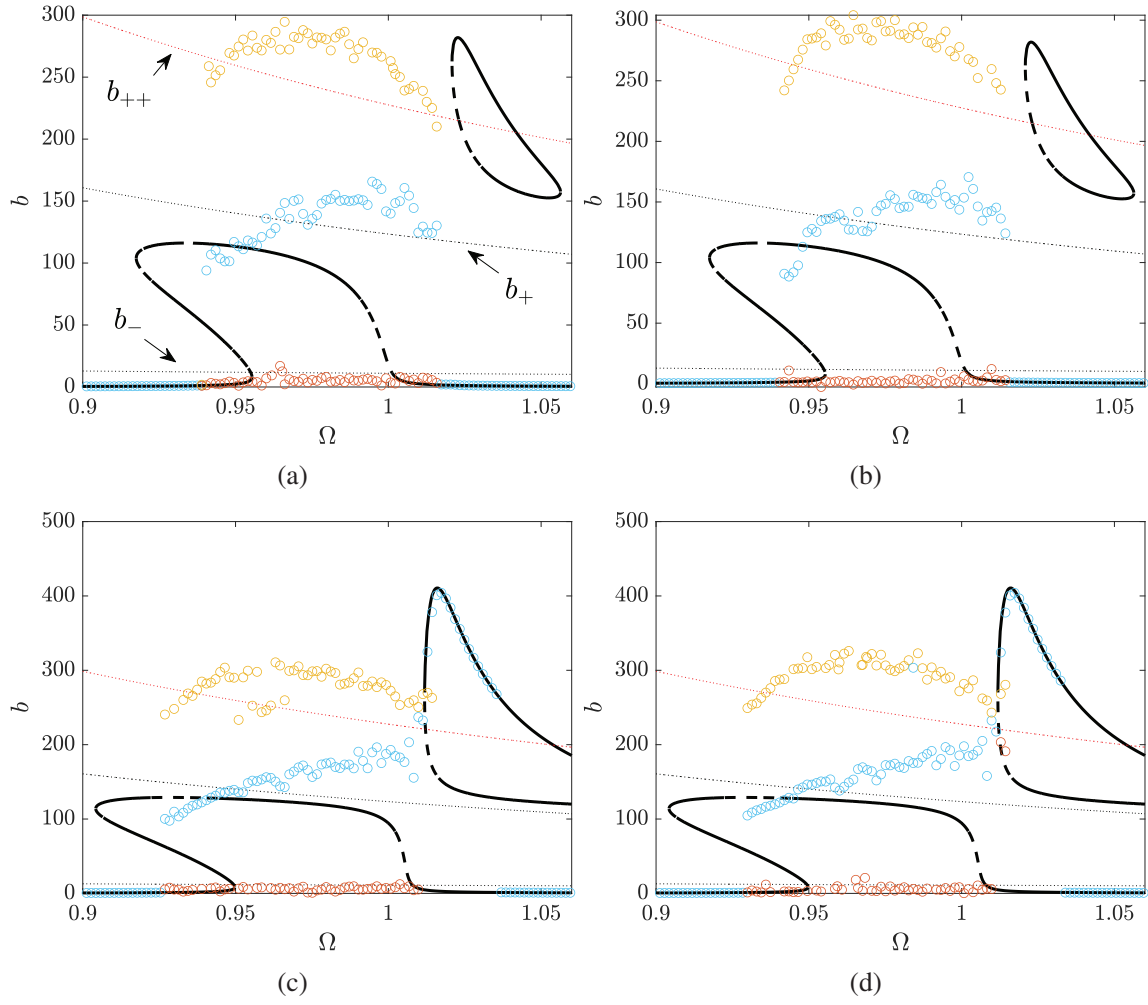


Figure 7: FR for b , left side is arctan and rightside saturated voltage. (a)(b) $\bar{F}_i = 20$ (c)(d) $\bar{F}_i = 25$

- [9] K. Dekemele and G. Habib, “Inverted resonance capture cascade: modal interactions of a nonlinear energy sink with softening stiffness,” *Nonlinear Dynamics*, pp. 1–23, 2023.
- [10] S. Zhang, J. Zhou, H. Ding, K. Wang, and D. Tan, “Theoretical investigation on vibration mitigation in a system with fractional nonlinear energy sinks,” *Nonlinear Dynamics*, pp. 1–23, 2024.
- [11] L. Manevitch, “The description of localized normal modes in a chain of nonlinear coupled oscillators using complex variables,” *Nonlinear Dynamics*, vol. 25, pp. 95–109, 2001.
- [12] L. Guillot, B. Cochelin, and C. Vergez, “A taylor series-based continuation method for solutions of dynamical systems,” *Nonlinear Dynamics*, vol. 98, no. 4, pp. 2827–2845, 2019.
- [13] L. Guillot, A. Lazarus, O. Thomas, C. Vergez, and B. Cochelin, “A purely frequency based floquet-hill formulation for the efficient stability computation of periodic solutions of ordinary differential systems,” *Journal of Computational Physics*, vol. 416, p. 109477, 2020.