

A Parametric Study on Magnetic Force Effects in Enhancing the Dynamic Response of Vibration Energy Harvester

Kishor Kumar, Amit Tyagi

Abstract

This paper presents an enhanced multi-stable vibration energy harvester (VEH) that integrates a cantilever beam with a tip magnet and three external magnets, operating under the influence of a nonlinear magnetic force. The magnetic force surrounding the harvester tip plays a critical role in influencing the harvester's output response. The magnetic force acting on the beam tip is effectively tuned by carefully selecting key parameters such as the relative gaps between the magnets, the size of the external magnets, and the inclination of the side magnets. This strategic combination of parameter adjustments aims to enhance the power output performance of the multi-stable harvester. The cantilever beam VEH is modelled using Euler-Bernoulli beam theory, and the governing equations of the VEH system are derived through the application of the Euler-Lagrange equations. The nonlinear dynamic behaviour of the VEH system is analysed under low-intensity vibrations of up to 1 g within a low-frequency range (below 22 Hz) using both the method of multiple scales and numerical simulations. The theoretical model is validated through experimental results, which demonstrate that an improved VEH design can be achieved by optimising these parameters. Furthermore, the effects of key parameters such as the relative gap, the volume ratio of external magnets, and the inclination angle of side magnets are comprehensively analysed through experimental studies.

Keywords: Vibration Energy Harvester (VEH), Nonlinear Dynamics, Method of Multiple Scales, Multi-stable.

Nomenclature

Subscripts h, p, a, b, c	host beam and piezoelectric beam, magnets A, B, and C
ρ, A, b, h, L	Density, cross-sectional area, width, thickness, length
$B_r, B_{j,a}, \mu_0$	Magnetic residual flux density and Magnetic flux density, Permeability of free space
C, ζ, ω_n	Damping coefficient, Damping ratio, Natural frequency of the harvester
C_{11}, Y	Elastic modulus of the piezoelectric and brass beam
$C_p, e_{31}, \epsilon_{33}$	Capacitance, Coupling constant, and Permittivity constant
$M, V, \vec{\mu}$	Magnetisation vectors, volume of magnets, and Magnetic moment vectors

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$w(x, t), q(t), v(t)$	Transverse deflection of the harvester beam, Generalised displacement coordinate, Voltage across resistance
R, θ	Load resistance, Coupling coefficient
$\vec{r}_{ab}, \vec{r}_{ac}$	Position vectors from magnet A to B and A to C
$r, s, s_1, \theta, \delta$	Vertical distance of magnet C from A, Horizontal distances of magnet B from A, and Horizontal distances of magnet C from A, Slope, inclination angle
$\ddot{y}_b(t)$	Base excitation acceleration

1 INTRODUCTION

Energy harvesting offers a sustainable solution for low-power devices, with vibration energy harvesting (VEH) gaining prominence for converting mechanical vibrations into electricity. Early VEH designs had limited efficiency beyond resonance, leading to innovations like multi-degree-of-freedom systems, frequency up-conversion, magnetic coupling, parametric and internal resonance, and self-tuning mechanisms.

Magnetic coupling is widely utilised in piezoelectric vibration energy harvesters (PVEHs) to modify stiffness and enhance performance. Research has demonstrated that integrating magnetic interactions significantly improves energy harvesting efficiency. Shahriz [1] introduced a magnetic proof mass at the tip of a cantilever beam with nearby attractive permanent magnets, showing that optimal magnetic strength and placement amplify tip vibrations. Repulsive magnetic configurations shift the beam's equilibrium, creating a bi-stable system with two equilibrium points. Stanton et al. [2] confirmed through numerical and experimental studies that bi-stable harvesters with repulsive magnets exhibit an expanded operational frequency range. Similarly, Ferrari et al. [3] demonstrated that coupling a piezoelectric beam converter with permanent magnets enables oscillations between stable states under random vibrations, improving energy harvesting. Al-Ashtari et al. [4] proposed a frequency self-tuning technique, showing that small changes in magnet distance alter the magnetic force, thereby influencing the generated voltage. Cao et al. [5] investigated the nonlinear dynamics of a piezoelectric energy harvester by incorporating external magnets at adjustable angles.

Additional magnets can create multi-stable systems with multiple equilibrium points, allowing harvesters to transition between them under sufficient excitation. Zhou et al. [6] introduced a tri-stable PVEH with three potential wells, demonstrating that tri-stable configurations enable easier snap-through transitions and higher power output over a wider frequency range than bi-stable systems. Leng et al. [7] investigated a tri-stable PVEH under random excitation, using equivalent magnetising current theory to model magnetic forces. Their findings emphasised the importance of optimising system parameters to facilitate large-amplitude inter-well motion, significantly enhancing output voltage. Zhou et al. [8] later introduced a quad-stable PVEH by adjusting three external magnets, showing that this configuration provides a lower threshold energy and a wider frequency bandwidth. Ju et al. [9] extended this concept to a quin-stable PVEH, combining magnetic nonlinearity with mechanical linearity to achieve broadband frequency operation.

Since the magnetic field directly affects the harvester's response, researchers have explored techniques to actively modify it. Chen et al. [10] investigated a bi-stable harvester with a self-decreasing potential barrier, incorporating a spring-magnetic oscillator to regulate magnetic force. Zhang et al. [11] proposed a multi-stable harvester utilising rotatable external magnets to alter potential well characteristics. Yang and Towfighian [12] explored internal resonance by integrating a movable magnet to actively adjust the magnetic field. Mei et al. [13] developed a magnetic-linkage PVEH for effective low-intensity excitation across a wide frequency range using time-varying potential wells. Lan and Qin [14] introduced an additional small magnet between two external magnets, lowering the energy barrier in a bi-stable harvester under random excitation and facilitating snap-through transitions.

The strength of the magnetic field between the tip magnet and external magnets is crucial in determining PVEH output response. Parameters such as the relative spacing, size, and inclination angle of magnets influence field strength. Existing literature has not comprehensively examined their independent effects. This study systematically analyses these parameters to optimise PVEH performance subjected to low-intensity vibrations. Using the Runge-Kutta method and the method of multiple scales, the nonlinear dynamic response of the PVEH is evaluated. Experimental validation confirms the theoretical model and highlights improvements in PVEH efficiency, providing valuable insights for optimising magnetic force parameters in energy harvesting applications.

2 MATHEMATICAL MODELLING OF THE PVEH SYSTEM

The PVEH system consists of a cantilever beam with a piezoelectric patch near its fixed end and a magnetic proof mass (A) at the tip, as shown in Figure 1. Three adjustable external magnets—middle (B) and side (C)—are positioned at varying distances (s , s_1 , r) from A. The study analyses how the relative positions, sizes, and orientations of magnets influence performance under low base excitation. Systematic testing across multiple configurations evaluates their combined impact on the harvester's output response.

The cantilever beam harvester is modelled using Euler-Bernoulli beam theory, with governing dynamic equations derived via Euler-Lagrange equations. The simplified electromechanical equations for the PVEH system, as derived in the author's previous work [15], are expressed as follows:

$$M \ddot{q}(t) + C \dot{q}(t) + K_1 q(t) + K_3 q^3(t) + K_5 q^5(t) + \theta V(t) + K_{12} q^2(t) V(t) + F_m = -m \ddot{y}_b(t) \quad (1)$$

$$\theta \dot{q}(t) + K_{12} q^2(t) \dot{q}(t) + \frac{V(t)}{R} + C_p \dot{V}(t) = 0 \quad (2)$$

M , C are equivalent mass and damping, and K represents the stiffness coefficient. The presence of nonlinear stiffness terms indicates that nonlinear strain effects are taken into account. The parameters K_{12} and θ characterise the electromechanical coupling within the system equations. The displacement and voltage output are represented by $q(t)$ and $V(t)$ respectively. C_p , R represent the capacitance of the piezoelectric material and resistive load whereas $\ddot{y}_b$ indicates the base excitation.

The magnetic potential energy and the nonlinear repulsive magnetic force F_m due to the interaction of permanent magnets are modelled using the point dipole approach [8], [14], [16].

The magnetic moment vectors for magnets A, B, and C are given in Eq. (1):

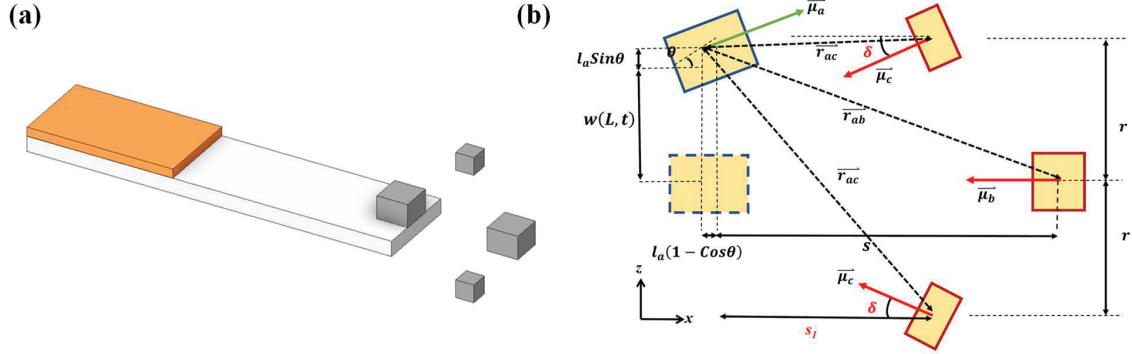


Figure 1: a The PVEH system with additional magnets in geometric configuration, **b** various design configurations of the tip magnet and external magnets

$$\left. \begin{aligned} \vec{\mu}_a &= \{M_a V_a \cos \theta, M_a V_a \sin \theta\} \\ \vec{\mu}_b &= \{-M_b V_b, 0\} \\ \vec{\mu}_c &= \{-M_c V_c \cos \delta, -M_c V_c \sin(-\delta)\} \end{aligned} \right\} \quad (3)$$

The position vectors given by $\vec{r}_{ab}$ and $\vec{r}_{ac}$, are expressed as:

$$\begin{aligned} \vec{r}_{ab} &= \{-s - l_a(1 - \cos \theta), w(L, t) + l_a \sin \theta\} \\ \vec{r}_{ac} &= \{-s_1 - l_a(1 - \cos \theta), w(L, t) + l_a \sin \theta \pm r\} \end{aligned} \quad (4)$$

The positive and negative signs indicate the position of side magnets C relative to magnet A. Slope is obtained as follows: $\theta = w'(L, t)$. Magnetisation vector M for the magnets is calculated by dividing the magnetic residual flux density, B_r , by the permeability of free space, μ_0 , as: $M = \frac{B_r}{\mu_0}$. The magnetic flux density generated by middle magnet B and side magnets C on magnet A is given by:

$$\vec{B}_{j,a} = \sum_{j=b,c} -\frac{\mu_0}{4\pi} \nabla \left(\frac{\vec{\mu}_j \cdot \vec{r}_{a,j}}{\|\vec{r}_{a,j}\|_2^3} \right) \quad (5)$$

The magnetic potential energy, U_m , of the specified magnetic configuration is expressed as follows:

$$U_m = \sum_{j=b,c} (-\vec{B}_{j,a} \cdot \vec{\mu}_a) \quad (6)$$

The total magnetic force $\vec{F}_m$ can be calculated as follows:

$$\vec{F}_m = -\nabla(U_m) = \sum_{j=b,c} -\frac{\mu_0}{4\pi} \nabla \left(\frac{\vec{\mu}_j}{\|\vec{r}_{a,j}\|_2^3} - \frac{3\vec{r}_{a,j}}{\|\vec{r}_{a,j}\|_2^5} (\vec{\mu}_j \cdot \vec{r}_{a,j}) \right) \vec{\mu}_a \quad (7)$$

The total vertical magnetic force due to all additional magnets on the tip magnet, F_m , is approximated by a polynomial expression.

$$F_m = \sum_n K_{nf} q^n(t) \quad (8)$$

Where $n = 1, 3, 5, 7, 9$, and 11 . The nonlinear magnetic force F_m is substituted in Eq. (1) to obtain the simplified model. The response of the dynamic system exhibiting nonlinearities can be obtained using the analytical approximate method of multiple scales (MMS). The system is defined by the following simplified equations with the introduction of ϵ as a book-keeping parameter.

$$\ddot{q}(t) + \omega_n^2 q(t) + \epsilon(2\omega_n \zeta \dot{q}(t) + k_3 q^3(t) + k_5 q^5(t) + \theta_1 V(t) + k_{12} q^2(t) V(t) + k_7 q^7(t) + k_9 q^9(t) + k_{11} q^{11}(t)) + (m/M)\ddot{y}_b(t) = 0$$

$$C_p \dot{V}(t) + \frac{V(t)}{R} + (\theta \dot{q}(t) + K_{12} q^2(t) \dot{q}(t)) = 0 \quad (9)$$

The time scale dependence of the solution is defined to keep the expansion uniformly valid up to $O(\epsilon^{-j})$ as $T_j = \epsilon^j t$. Two-term approximation of the response solution and the time derivatives obtained are given using the following expressions:

$$\begin{aligned} q(t) &= q_0(T_0, T_1) + \epsilon q_1(T_0, T_1) \\ V(t) &= V_0(T_0, T_1) + \epsilon V_1(T_0, T_1) \end{aligned} \quad (10)$$

$$\frac{d}{dt} = \frac{dT_0}{dt} \frac{\partial}{\partial T_0} + \frac{dT_1}{dt} \frac{\partial}{\partial T_1} + \dots = D_0 + \epsilon D_1 + O(\epsilon^2)$$

$$\frac{d^2}{dt^2} = \frac{d^2}{dT_0^2} + 2\epsilon \frac{d^2}{dT_0 dT_1} + \dots = D_0^2 + 2\epsilon D_0 D_1 + O(\epsilon^2) \quad (11)$$

After substituting the solutions from Eq. (10) in Eq. (9) and collecting the coefficients of ϵ , the following equations are obtained.

$$\begin{aligned} O(\epsilon^0): \quad & \frac{\partial^2 q_0}{\partial T_0^2} + \omega_n^2 q_0 = 0 \\ & C_p \frac{\partial V_0}{\partial T_0} + \frac{V_0}{R} = -(\theta \dot{q}_0 + K_{12} q_0^2 \dot{q}_0) \end{aligned} \quad (12)$$

$O(\epsilon^1)$:

$$\frac{\partial^2 q_1}{\partial T_0^2} + \omega_n^2 q_1 = 2\zeta \omega_n \frac{\partial q_0}{\partial T_0} + k_3 q_0^3 + k_5 q_0^5 + k_7 q_0^7 + k_9 q_0^9 + k_{11} q_0^{11} + k_{12} q_0^2 V_0 + 2 \frac{\partial^2 q_0}{\partial T_0 \partial T_1} + \theta_1 V_0 + f \quad (13)$$

$$C_p \frac{\partial V_1}{\partial T_0} + \frac{V_1}{R} = C_p \frac{\partial V_0}{\partial T_1} + 2K_{12} \frac{\partial q_0}{\partial T_0} q_0 q_1 + K_{12} \frac{\partial q_0}{\partial T_1} q_0^2 + K_{12} q_0^2 \frac{\partial q_1}{\partial T_0} + \theta \frac{\partial q_0}{\partial T_1} + \theta \frac{\partial q_1}{\partial T_0} \quad (14)$$

The solution of zero-order perturbation given in Eq. (12) can be written as:

$$q_0 = A(T_1) e^{iT_0 \omega_n} + \bar{A}(T_1) e^{-iT_0 \omega_n} \quad (15)$$

$$V_0 = -\frac{e^{iT_0 \omega_n} R \theta A(T_1) \omega_n}{-i + RC_p \omega_n} - \frac{e^{3iT_0 \omega_n} R K_{12} A^3(T_1) \omega_n}{-i + 3RC_p \omega_n} - \frac{e^{iT_0 \omega_n} R K_{12} A^2(T_1) \omega_n \bar{A}(T_1)}{-i + RC_p \omega_n} + e^{-\frac{T_0}{RC_p}} + cc \quad (16)$$

$$\omega = \omega_n + \epsilon \sigma$$

Where cc represents the complex conjugate of preceding terms and σ represents the detuning parameter. The solutions obtained in Eq. (15) are substituted in Eq. (13) for the solution of the first order. To obtain the bounded solution, the resulting secular terms are eliminated by setting the

coefficient of $e^{-iT_0\omega_n}$ to zero. The solution of amplitude, $A(T_1)$, is obtained by expressing it in polar form as:

$$\begin{aligned} A(T_1) &= \frac{1}{2}a(T_1)e^{i\phi_1(T_1)} \\ \bar{A}(T_1) &= \frac{1}{2}a(T_1)e^{-i\phi_1(T_1)} \end{aligned} \quad (17)$$

Here $a(T_1)$ and $\phi_1(T_1)$ represent the amplitude and phase. To obtain the solution, the real and imaginary parts of the equations are set to zero. Further simplification of these equations is done by rearranging the terms.

$$\frac{1}{2}f \cos(\gamma(T_1)) + a(T_1)H_7 + a(T_1)^3H_8 + a(T_1)^5H_9 = \omega_n \frac{\partial a(T_1)}{\partial T_1} \quad (18)$$

$$\begin{aligned} &\frac{1}{2}f \sin(\gamma(T_1)) + H_1a(T_1) - a(T_1)\omega_n\sigma + a(T_1)^3H_2 + a(T_1)^5H_3 + a(T_1)^7H_4 + H_5a(T_1)^9 + a(T_1)^{11}H_6 = \\ &-a(T_1)\omega_n \frac{\partial \gamma(T_1)}{\partial T_1} \end{aligned} \quad (19)$$

Where:

$$\gamma(T_1) = T_1\sigma - \phi_1(T_1), G_1 = \frac{R}{R^2C_p^2\omega_n^2+1}, G_2 = \frac{R}{9R^2C_p^2\omega_n^2+1}, H_1 = -\frac{1}{2}(G_1\theta\theta_1RC_p\omega_n^2),$$

$$H_2 = (-\frac{G_1R\theta C_p k_{12}\omega_n^2}{4} + \frac{3}{8}k_3 - \frac{G_1RC_p K_{12}\theta_1\omega_n^2}{8}), H_3 = (-\frac{G_1RC_p k_{12}K_{12}\omega_n^2}{16} + \frac{5}{16}k_5 - \frac{3G_2RC_p k_{12}K_{12}\omega_n^2}{32}), H_4 = \frac{35}{128}k_7,$$

$$H_5 = \frac{63k_9}{256}, H_6 = \frac{231k_{11}}{1024}, H_7 = (-\zeta\omega_n^2 + \frac{1}{2}G_1\theta\theta_1\omega_n), H_8 = \frac{1}{4}G_1\theta k_{12}\omega_n + \frac{1}{8}G_1\theta_1 K_{12}\omega_n,$$

$$H_9 = \frac{1}{16}G_1 k_{12}K_{12}\omega_n + \frac{1}{32}G_2 k_{12}K_{12}\omega_n.$$

To obtain the steady state solution, the terms in Eqs. (18) and (19) are squared and added, then the derivatives of amplitude and phase with respect to time are set to zero, $\frac{\partial a(T_1)}{\partial T_1} = \frac{\partial \gamma(T_1)}{\partial T_1} = 0$. The resulting equation is a polynomial equation of order 11 in a and order 2 in σ .

$$(aH_1 + a^3H_2 + a^5H_3 + a^7H_4 + a^9H_5 + a^{11}H_6 - a\omega_n(\sigma))^2 + (aH_7 + a^3H_8 + a^5H_9)^2 = \frac{1}{2}f^2 \quad (20)$$

The solution of the above equation provides the amplitude and phase of the response solution. To get the stability analysis of the PVEH system, the Jacobian matrix is calculated as follows:

$$J(a, \sigma) = \begin{bmatrix} \frac{H_7+3a^2H_8+5a^4H_9}{\omega_n} & \frac{aH_1+a^3H_2+a^5H_3+a^7H_4+a^9H_5+a^{11}H_6-a\sigma\omega_n}{\omega_n} \\ -\frac{H_1+3a^2H_2+5a^4H_3+7a^6H_4+9a^8H_5+11a^{10}H_6-\sigma\omega_n}{a\omega_n} & \frac{aH_7+a^3H_8+a^5H_9}{a\omega_n} \end{bmatrix} \quad (21)$$

The Jacobian matrix's eigenvalues determine PVEH system stability and bifurcations. The solution is stable if all eigenvalues have negative real parts and unstable if any have a positive real part. These conditions are crucial for analysing bifurcations, where parameter changes alter system response qualitatively. The steady state solution amplitude and voltage can be calculated using Eqs. (15) and (16).

3 THEORETICAL INVESTIGATIONS

Theoretical analysis examines the total potential energy and magnetic force to assess the impact of different geometric configurations. The material properties of the harvester system are listed in Table 1.

Table 1: Material Properties

Parameter	Value
Brass beam properties	
$L \times b_h \times h_h$ (mm ³)	130 x 20 x 0.5
Elastic modulus of brass beam Y (GPa)	100
Density of brass beam ρ_h (Kg/m ³)	8570
MFC properties	
$L_p \times b_p \times h_p$ (mm ³)	28 x 14 x 0.3
Elastic modulus C_{11} (GPa)	30.336
Density of MFC beam ρ_p (Kg/m ³)	5440
Coupling constant e_{31} (C/m ²)	-5.16
Permittivity constant ϵ_{33} (F/m)	9.18×10^{-8}
Magnetic properties	
Tip mass M_{tip} (grams)	7.474
Residual magnetic flux density B_r (T)	1.25
Permeability of free space μ_0 (H/m)	$4\pi \times 10^{-7}$

Figure 2 illustrates the potential energy landscape and corresponding magnetic force variations for different system configurations. The first and third rows depict the potential energy distribution, while the second and fourth rows represent the associated nonlinear magnetic force. Figures 2a and 2b analyse the effect of horizontal distance s in the presence of a single middle magnet, $V_b = 1000 \text{ mm}^3$. As s decreases from 25 mm to 15 mm, the nonlinear magnetic force intensifies, reaching up to 5 N, strengthening bi-stability. However, this increased force makes inter-well transitions more difficult, requiring higher excitation. Figures 2c and 2d examine the effect of horizontal distances s and s_1 when three external magnets, $V_b = V_c = 1000 \text{ mm}^3$, symmetrically surround the tip magnet. As these distances decrease, additional potential wells emerge, and existing wells deepen, with magnetic force reaching approximately 5 N. Figures 2e and 2f investigate the impact of vertical distance r . Reducing r alters the potential well depth and weakens the magnetic force, which remains within 2 N. Figures 2g and 2h investigate the effect of the horizontal distance of side magnets s_1 . Decreasing this distance leads to the formation of multiple potential wells, with deeper outer wells at higher potential energy and shallower inner wells at lower potential energy. The magnetic force exhibits weaker nonlinearity at lower amplitudes but becomes strongly nonlinear at higher amplitudes, reaching up to 5 N. Figures 2i and 2j explore the effect of side magnet size. With the middle magnet volume fixed at 648 mm^3 , increasing side magnet volume from 50 mm^3 to 1800 mm^3 transitions the system from bi-stable to quad-stable, increasing potential well depth and magnetic force up to 5 N. Figures 2k and 2l assess inclination angle effects while $V_b = V_c = 1000 \text{ mm}^3$. Adjusting the angle reshapes potential wells, transitioning the system between bi-stable and multi-stable states. The magnetic force remains below 4 N.

Overall, Figure 2 highlights the crucial role of geometric parameters in shaping energy landscapes and nonlinear interactions, emphasising the importance of tuning these parameters to optimise energy harvesting performance.

Figure 3 presents the potential energy and corresponding magnetic force as a function of amplitude for three different configurations of the PVEH system. The left column shows the potential energy, while the right column illustrates the magnetic force with respect to amplitude.

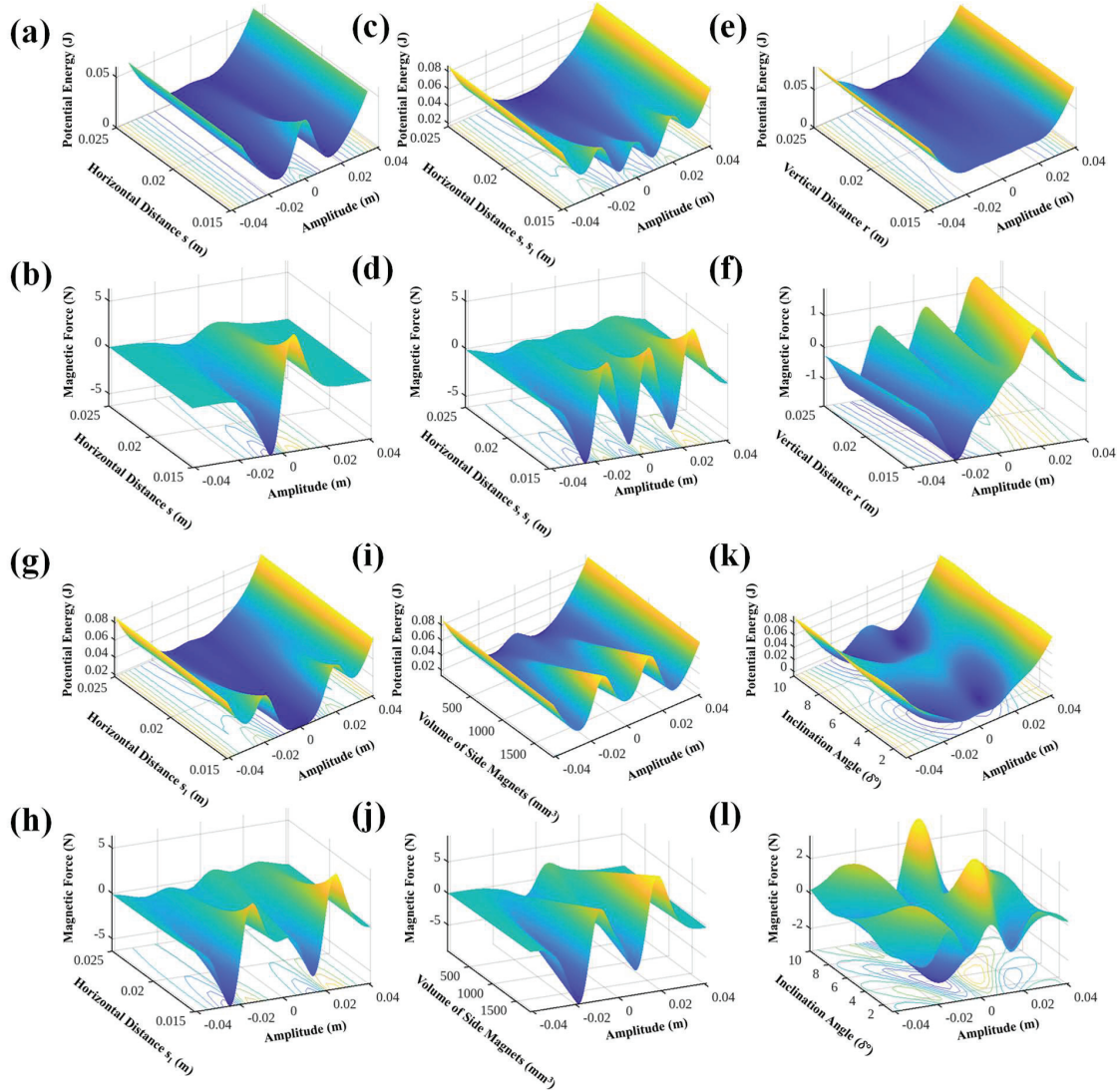


Figure 2: Potential energy and magnetic force with respect to amplitude for configurations **a** bi-stable **c** $r = 22$ mm **e** $s = s_1 = 22$ mm **g** $s = 22$ and $r = 22$ mm **i** $s = 17$, $r = 15$, $s_1 = 16$ mm, and $V_b = 648$ mm³ **k** $s = 21$, $r = 18$, and $s_1 = 21$ mm **b, d, f, h, j, l** corresponding magnetic force respectively

Figure 3a shows the potential energy profile exhibits a quad-stable behaviour with multiple wells, indicating that the system can operate under multi-stable conditions for configurations $s = s_1 = 15$

mm and $r = 22$ mm. The corresponding magnetic force shows an alternating pattern, which suggests strong nonlinear interactions between the magnets. The presence of multiple potential wells allows for large tip displacements under dynamic excitation, which can enhance power generation under sufficient excitation. Figure 3b illustrates that the potential energy profile is relatively shallow and symmetric, suggesting a system with reduced nonlinearity and limited multi-stable behaviour for configurations $s = s_1 = 22$ mm and $r = 15$ mm. The magnetic force exhibits a smoother variation with amplitude, implying a weaker magnetic influence. Figure 3c indicates the potential energy profile is characterised by a bi-stable nature, with two dominant wells for configurations $s = s_1 = 22$ mm and $r = 22$ mm. The magnetic force varies gradually, maintaining a moderate level of nonlinearity.

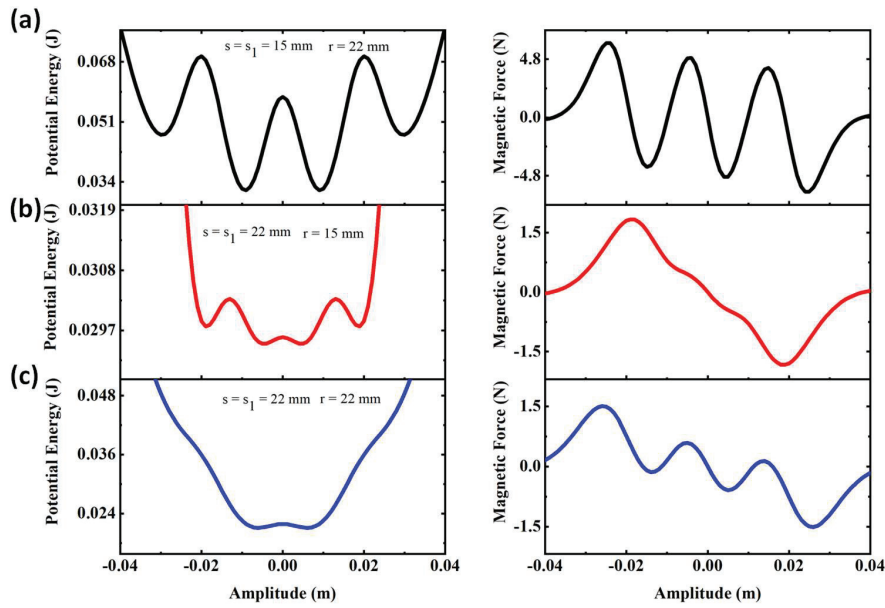


Figure 3: Potential energy function and magnetic force under different configuration types

Nonlinear magnetic interactions play a crucial role in shaping the potential energy landscape and influencing the harvester's performance. For maximising energy harvesting performance, configurations with shallower and multiple potential wells are preferred. These conditions allow the harvester to transition between states efficiently, utilising nonlinear dynamic effects to generate higher power output.

Figure 4 presents the nonlinear frequency response of the system, comparing theoretical predictions with experimental results under 5 m/s^2 excitation. The plots show tip displacement amplitude versus excitation frequency, demonstrating multi-stable behaviour. The black solid line represents the stable solution branch, while the red dashed line indicates the unstable branch, separated by a saddle-node (SN) bifurcation. Blue dots denote experimental data, highlighting real-world nonlinear effects. Quantitative deviations arise from factors such as structural damping, fabrication imperfections, and environmental noise. The observed jump phenomena, characteristic of nonlinear systems, facilitate inter-well motion, enhancing tip displacement and improving power output. The results confirm that tuning system parameters significantly impact stability, response amplitude, and overall energy harvesting efficiency.

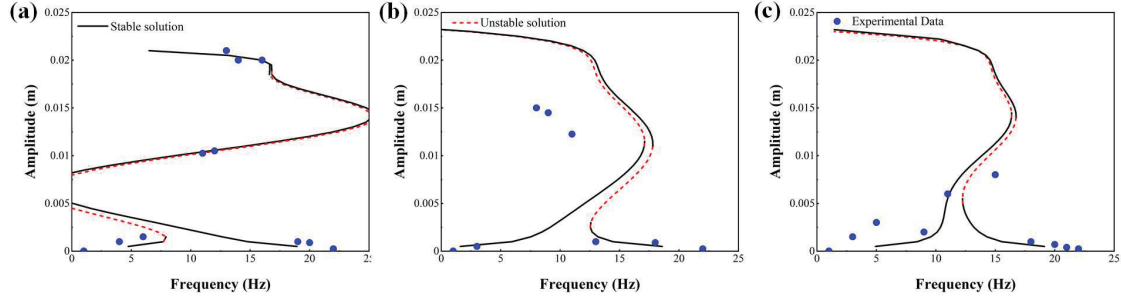


Figure 4: Amplitude vs frequency results obtained by MMS and experimental data for configuration **a** $s = s_1 = 15$ mm and $r = 22$ mm, **b** $s = s_1 = 22$ mm and $r = 15$ mm, **c** $s = s_1 = 22$ mm and $r = 22$ mm

4 EXPERIMENTAL INVESTIGATIONS

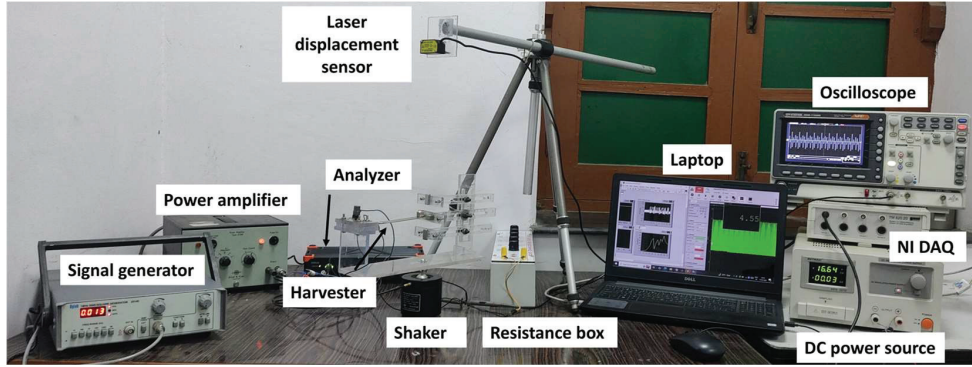


Figure 5: Experimental setup for PVEH

The experimental setup is designed to analyse the output response of a piezoelectric vibration energy harvester (PVEH) system, Figure 5. A signal generator produces an excitation signal, which is amplified using a power amplifier and fed to a shaker that induces controlled vibrations in the harvester. The harvester converts mechanical vibrations into electrical energy, and its response is analysed using multiple instruments. A laser displacement sensor measures the tip displacement, while an oscilloscope captures the voltage output. A resistance box provides a variable resistive load to study power output under different conditions. The NI DAQ collects and transfers experimental data to a laptop, where real-time visualisation and analysis are performed.

5 RESULTS AND DISCUSSION

Figure 6 presents the power output, time response, and phase portraits of the PVEH system under different magnet configurations. The first column shows power output (in mW) as a function of excitation frequency (Hz) for various horizontal (s , s_1) and vertical (r) distances under a $400 \text{ k}\Omega$ external load. The results indicate that smaller horizontal distances enhance magnetic coupling, leading to higher power generation. The configuration $s = s_1 = 15$ mm achieves the highest power ($\sim 0.27 \text{ mW}$) around $15\text{--}18 \text{ Hz}$ for $r = 22$ mm. Increasing s and s_1 (to 18 mm , 22 mm , and 25 mm) reduces peak power but broadens the frequency response. The optimal configuration for $r = 18 \text{ mm}$ ($s = s_1 = 22 \text{ mm}$) produces the highest peak power ($\sim 0.30 \text{ mW}$) near $12\text{--}15 \text{ Hz}$. Lower values of r shift resonance to lower frequencies ($\sim 10 \text{ Hz}$), emphasising the role of magnetic interactions in

tuning response. The configuration $s_1 = 25$ mm with $s = r = 22$ mm achieves the highest power (~ 0.27 mW) at 15 Hz, outperforming other setups.

The second and third columns present the experimental time response and phase portraits compared with mathematical simulations. The time response for $s = s_1 = 15$ mm, $r = 22$ mm (Figure 6b) exhibits an initial oscillatory motion followed by chaotic behaviour, confirmed by the quad-stable trajectories in Figure 6c. Figure 6e shows intra-well oscillations for $s = s_1 = 22$ mm, $r = 15$ mm, indicating higher excitation requirements for inter-well transitions. The phase portrait (Figure 6f) reveals double-well oscillations in shallow wells. Figure 6h exhibits more stable oscillations for $s = s_1 = 22$ mm, $r = 22$ mm, with phase confinement to a single well, indicative of a bi-stable response.

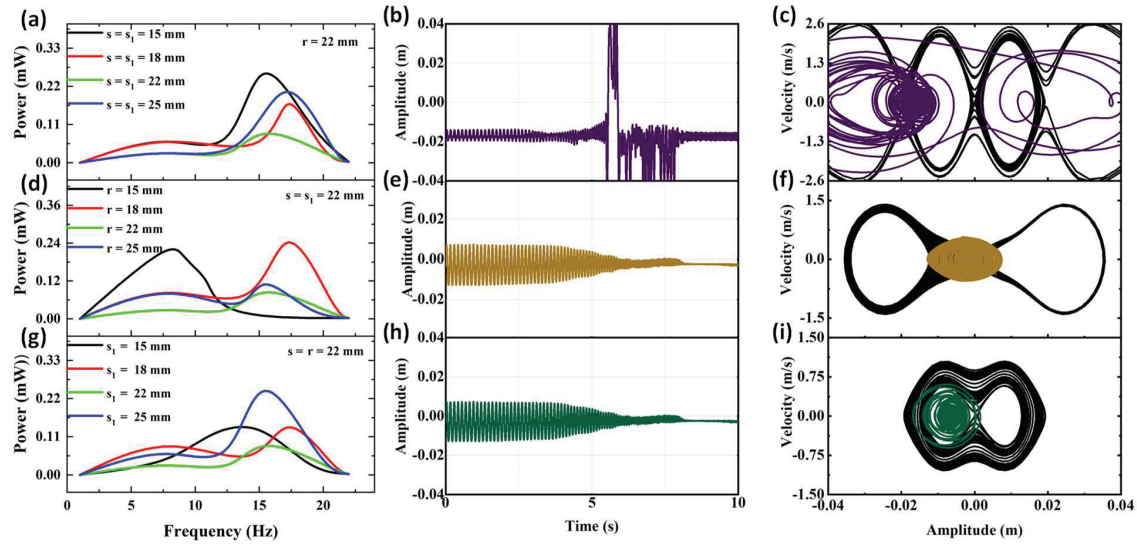


Figure 6: **a** Power vs frequency, **b** displacement amplitude vs time, **c** phase portraits

Overall, Figure 6 captures quad-stable and bi-stable regimes depending on system parameters, highlighting the nonlinear characteristics of the harvester. Reducing the horizontal distance (s, s_1) enhances peak power output but limits operational frequency range. Optimising vertical distance achieves maximum power. A combination of smaller horizontal distances ' s ', moderate vertical spacing ' r ', and larger horizontal distance ' s_1 ' provides the most efficient energy harvesting configuration.

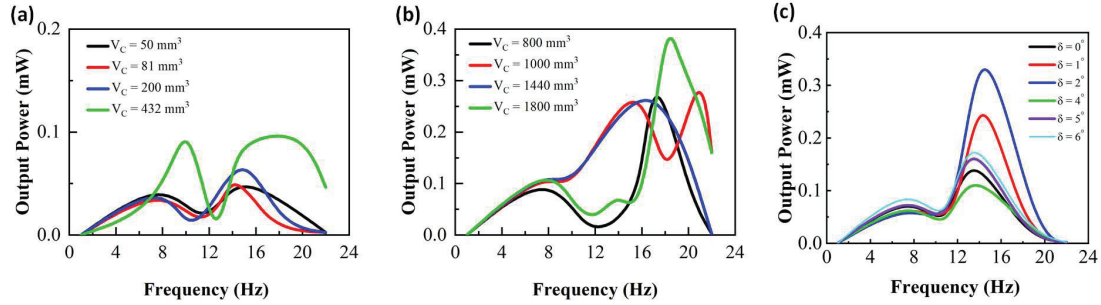


Figure 7: a and b Power vs frequency under for a different size of external magnets for relative position $s = 17$, $r = 15$, and $s_1 = 16$ mm and $V_b = 648$ mm³ c under different inclination angle for relative position $s = 21$, $r = 18$, and $s_1 = 21$ mm and $V_b = V_c = 1000$ mm³

Figure 7 illustrates the effect of magnet volume and side magnet inclination on the harvester's output under an upward frequency sweep with a 400 k Ω load. The setup includes a middle magnet at 17 mm (horizontal) and side magnets at 15 mm (vertical) and 16 mm (horizontal). Figure 7a presents power output when side magnets are smaller than the middle magnet ($V_b = 648$ mm³), while Figure 7b shows results for larger side magnets. With smaller side magnets, both frequency response peaks remain nearly equal. However, larger side magnets significantly amplify the secondary peak, enhancing overall performance. Figure 7c examines the effect of inclination angle with the middle magnet at 21 mm (horizontal) and the side magnets at 18 mm (vertical) and 21 mm (horizontal). Increasing the inclination from 0° broadens the frequency range, with the secondary peak (~15 Hz) becoming more pronounced while the primary peak (~8 Hz) remains weaker. The optimal inclination (2°) yields peak power exceeding 0.3 mW between 13–19 Hz. Beyond this, power output initially decreases before increasing again, aligning with potential energy trends where magnet orientation cyclically affects potential well formation.

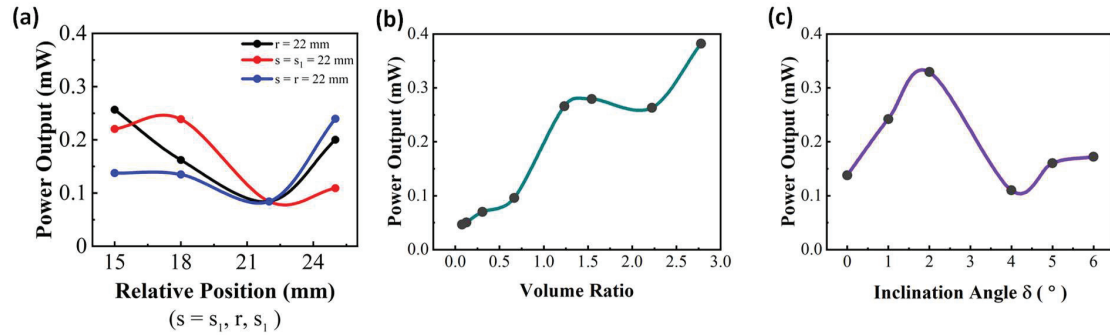


Figure 8: power output vs: a relative position b volume ratio c inclination angle

Figure 8 presents the influence of relative positions (s , r , and s_1), magnet volume ratio (side magnet volume/ middle magnet volume), and inclination angle on the power output of the PVEH system under a 400 k Ω load. Figure 8a shows power output variations for three configuration cases: (i) for $r = 22$ mm (black curve), (ii) for $s = s_1 = 22$ mm (red curve), and (iii) for $s = r = 22$ mm (blue curve) where the relative position represents the horizontal distances ($s = s_1$), vertical distance (r), and horizontal distance of side magnets (s_1) respectively. Initially, at 15 mm, all configurations achieve relatively high power, with the $r = 22$ mm configuration reaching ~0.25 mW. As the

relative position increases toward 21 mm, power output declines to ~ 0.1 mW for all configurations. Beyond 21 mm, power output rises again. This trend highlights the critical role of magnet positioning in optimising energy harvesting efficiency. Figure 8b explores peak power output as a function of magnet volume ratio for a fixed relative position ($s=17$ mm, $r=15$ mm, $s_1=16$ mm). As the volume ratio increases, peak power also rises. When the ratio exceeds 1, power jumps beyond 0.25 mW and continues increasing, approaching 0.4 mW. Figure 8c examines the effect of inclination angle on peak power for a different relative position ($s=21$ mm, $r=18$ mm, $s_1=18$ mm). Power initially increases with inclination, reaches a peak, then declines before rising again. This cyclical trend aligns with potential energy variations, demonstrating how inclination influences energy well formation and harvesting performance.

7 CONCLUSIONS

The analysis confirms that magnet positioning, volume ratio, and inclination angle are critical parameters influencing the power output of the PVEH system. Optimal relative placement of magnets enhances energy transfer and improves overall harvesting performance. Similarly, increasing the side-to-middle magnet volume ratio significantly boosts peak power, exceeding 0.25 mW, while appropriate inclination angles further optimise efficiency by reshaping the potential wells.

In conclusion, the findings emphasise the importance of parameter tuning in optimising the harvester's response for improved energy harvesting.

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