

Singularity Detection by K-Space Identification in Plate-like-Structures - Location Index Evaluation

Thomas Brion, Xi Li, Pascal Fossat, Mohamed Ichchou, Olivier Bareille, Abdelmalek Zine

Abstract. This paper presents a method for identifying singularity locations in plate-like structures. The proposed method is based on Algebraic K-Space Identification 2D (AKSI 2D) technique. This method has been validated numerically and experimentally, and its performances have been assessed with respect to other popular k-space identification methods under different uncertainty conditions.

Here, under the algebraic identification framework, the extraction of all properties of the complex k-space in a coherent manner is used for comparison purpose between a reference and disturbed vibrational field. The real part and the imaginary parts convey specific information about the vibrational field disturbances.

A specific subdomain strategy is employed in order to identify the precise location of singularities in plate-like elements. Results are presented on orthotropic plates. The procedure is able to locate singularities. It exhibits that the imaginary part of the identified k-space components gives better contrasted images of the singularities' locations.

Key words: k-space, plate structures, location index, vibration-based technique, SHM

1 INTRODUCTION

This paper is structured as follows. Section 2 presents the formula of the AKSI 2D. Then, Section 3 shows the validation of the proposed method on analytical fields on an example of numerical fields from a homogenised orthotropic 2D metastructure, namely a bi-ribbed plate.

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solved by a finite element method. Section 4 presents the specific strategy which is employed to identify the precise location of singularities. Results are presented on orthotropic plates, before the conclusion of the paper in section 5.

2 ALGEBRAIC K-SPACE IDENTIFICATION 2D: AKSI 2D FORMULAS

In order to later perform the analysis of singularity locations, the Algebraic K-Space Identification technique needs to be thoroughly presented. This is the purpose of this section with a specific emphasis on two-dimensional wavenumber calculations. The theory behind them is mostly inspired by the 1D method proposed by Li et al. [1].

Let U be a displacement field on a two-dimensional bounded domain Ω . In particular, $\Omega \subset [0, L_x] \times [0, L_y]$, $L_x, L_y \in \mathbb{R}^+$. The field is supposed to be known on some points of a subdomain Ω_w of Ω . The simplest case is when the field is known on a regular rectangular grid on Ω_w : (x_i, y_i) where i ranges from 1 to N_x and j ranges from 1 to N_y . From this displacement field, the AKSI 2D method provides a calculation of the wavenumbers in different directions. The first step is to calculate the coefficients γ by solving the system given by Eqn. (1) for numerous pairs of couples of the grid (x_0, y_0) and x, y such that $x_0 < x$ and $y_0 < y$.

The system is solved by a least squares method. The formula of the coefficients ϕ_{ik} needed in this equation is presented in Eqn. (2). In these formulas, n_w is the number of plane waves in the field Eqn. (4). The calculation of ϕ_{ik} needs integration calculations that can be done numerically, such as using the trapezoidal method.

$$\sum_{i=0}^{n_w} \sum_{k=0}^{n_w} \gamma(i, k) \phi_{ik}(x_0, y_0, x, y) = 0, \quad (1)$$

$$\phi_{ik}(x_0, y_0, x, y) = \sum_{j=0}^{n_w} \sum_{l=0}^{n_w} \frac{(-1)^{j+l} (n_w + j)! (n_w + l)! (x - x_0)^{n_w-j} (y - y_0)^{n_w-l}}{j! (i + j)! (n_w - j)! l! (k + l)! (n_w - l)!} \times \int_{y_0}^y \int_{x_0}^x (y - y_1)^{k+l} (x - x_1)^{i+j} U(x_1, y_1) dx_1 dy_1. \quad (2)$$

From the knowledge of the values of the coefficient γ , the wavenumber is calculated as a root of a defined polynomial equation Eqn. (3). For each direction θ , the wavenumber calculation requires a resolution of Eqn. (3) but with the same coefficients γ .

$$\Psi(K_\theta) = \sum_{l=0}^{2n_w} (-iK_\theta)^l \left(\sum_{m=\max(0, l-n_w)}^{\min(l, n_w)} \gamma(n_w - l + m, n_w - m) \cos^{l-m} \theta \sin^m \theta \right). \quad (3)$$

3 ANALYTICAL AND NUMERICAL VALIDATIONS

The method is now tested on fields calculated analytically and numerically. This method offers an interesting alternative to popular methods like the IWC [2] and the INCOME [3]. In order to illustrate the performance of the here presented method, an example of an orthotropic plate is used. Further analysis about its efficiency with respect to IWC and INCOME as well as about the robustness to noise and uncertainties can be found in the reference paper from Brion et al. [4]. By the way the following example is taken from it.

3.1 Numerical model of a bidirectional ribbed plate

A bidirectional ribbed plate is a periodic structure, illustrated in Fig. 1, made of a plate to which two perpendicular rows of stiffeners have been added. Since the beam grid of the stiffeners is supposed to be stiffer than the plates put between them, the dynamics of the stiffeners is dominant in the structure. That is why this metastructure presents a multiscale dynamic. The AKSI 2D method is tested on this type of structure to verify its efficiency on complex structures. Boutin et al. [5] have provided an analytical model of the flexural behavior of this type of structure that will serve as a reference. It is a model of an equivalent orthotropic plate obtained from an asymptotic homogenization process.

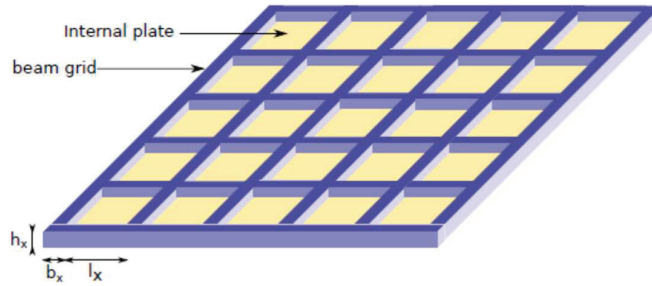


Figure 1: Example of a bidirectional ribbed plate of 5 by 5 cells [4]

This model is a 4th-order partial differential equation which result from the enrichment of an orthotropic plate equivalent to a beam grid, as the stiffeners are dominant compared to the plates in the global dynamics. Let h_j and b_j be respectively the height and the width of the stiffeners of the direction j . j being x or y . l_j is the contact length of the stiffeners with an inner plate. About the mechanical properties of the stiffeners: E_j , ν_j and ρ_j are respectively the Young Modulus, the Poisson ratio and the volume mass. A damping is added as an imaginary part of the Young modulus. η is the damping ratio and $\tilde{E}_j = E_j(1 + i\eta)$. From these parameters, some important quantities can be deduced: the bending inertia I_j , the torsional modulus G_j , the torsional inertia $\mathcal{I}_j$, the linear mass Λ_j and the torsional constant, $\rho_j J_j$.

In addition to these parameters, the properties of the internal plates must be defined. All the

plates have the same properties and are clamped to the beam grid. l_x, l_y are the side lengths and d is the width. E_p is the Young modulus, ν_p the Poisson ratio and ρ_p the volume mass. From these parameters, the following ones are calculated: the surface mass $\Lambda_p' = \rho_p d$, the bending inertia $I_p = d^3/12$ and the modified Young modulus $E_p' = E_p(1 + i\eta)/(1 - \nu_p^2)$. With these parameters, the equation of the model of an orthotropic plate equivalent to a bi-ribbed plate proposed is Eqn. (4).

$$\begin{aligned} & \frac{\tilde{E}_x I_x}{l_y} \frac{\partial^4 u}{\partial x^4} + \frac{\tilde{E}_y I_y}{l_x} \frac{\partial^4 u}{\partial y^4} + \left(\frac{G_x I_x}{l_y} + \frac{G_y I_y}{l_x} \right) \frac{\partial^4 u}{\partial x^2 \partial y^2} + \\ & \omega^2 \left(\frac{\rho_y J_y}{l_x} \frac{\partial^2 u}{\partial x^2} + \frac{\rho_x J_x}{l_y} \frac{\partial^2 u}{\partial y^2} \right) + \left(K_{\omega_x}^{bc} \frac{\partial u}{\partial x} + K_{\omega_y}^{bc} \frac{\partial u}{\partial y} \right) = \\ & \omega^2 \left(\frac{\Lambda_x}{l_y} + \frac{\Lambda_y}{l_x} + \Lambda_p' \langle \phi_{\omega}^{bc} \rangle \right) u. \end{aligned} \quad (4)$$

From Eqn. (4) the analytical dispersion relation is defined by replacing the solution u by a plane wave in the direction θ , $U(x, y)$.

3.2 Numerical simulation and comparison

The stiffeners are supposed to be the same in the x and the y directions and made of aluminum and the plates are made of plexiglass. In addition, a damping of $\eta = 5\%$ is added to the Young modulus as an imaginary part. Table 1 contains the properties of the stiffener and Table 2 contains those of the internal plate.

Table 1: Geometrical and mechanical properties of the beam grid

Parameter	h_x, h_y	b_x, b_y	E_x, E_y	ν_x, ν_y	ρ_x, ρ_y
Value	0.01m	0.01m	69GPa	0.3	2700kg/m ³

Table 2: Geometrical and mechanical properties of the internal plates

Parameter	l_x, l_y	d	E_p	ν_p	ρ_p
Value	0.1m	0.1m	3GPa	0.3	1200kg/m ³

A structure of 20 by 20 cells is considered, with a total side length of $L = 2.21$ m. The extremities are clamped, and a point load is applied at the center of the structure. The displacement fields are calculated by solving the analytical equation of the homogenized equivalent orthotropic plate model Eqn. (4) by finite element software. The fields are known on a grid of 91 by 91 regularly spaced points located on a window of side length $L_w = 1.808$ m centred on

the entire domain. The frequencies range from 5Hz to 500Hz. The AKSI 2D method is applied to these fields and the extracted wavenumbers are compared with the analytical model. The results extracted by the AKSI 2D are shown in Fig. 2. This figure shows that the wavenumbers are well calculated and match, those of the equivalent orthotropic plate model for most frequencies and directions. The inner resonance at 275Hz is even very well retrieved for both the real and imaginary parts.

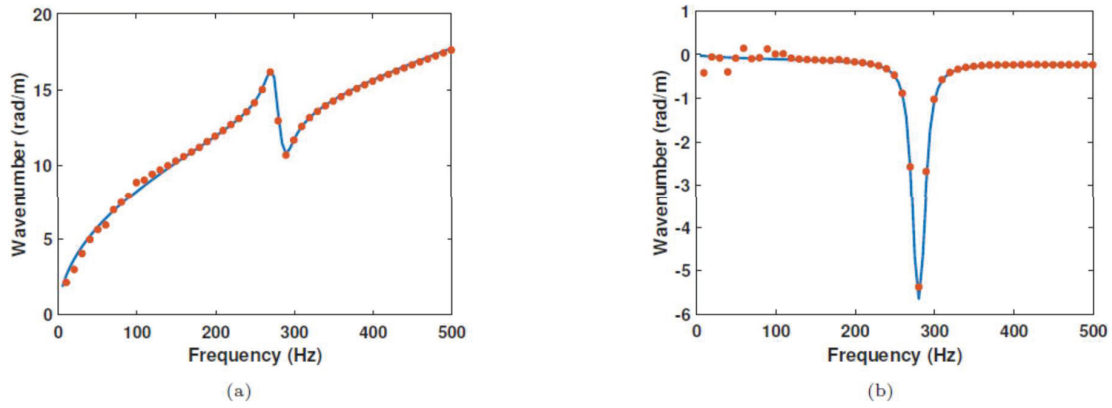


Figure 2: Dispersion relation in the direction 0 rad of the bi-ribbed plate: (a) real part, (b) imaginary part. Analytical model (solid line) and wavenumber calculated with the AKSI 2D (red dots) [4]

4 IDENTIFICATION OF SINGULARITY LOCATION: SUBDOMAIN STRATEGY

This section 4 presents specific subdomain strategy, which is employed in order to identify the precise location of singularities in plate-like elements. Results are presented on orthotropic plates and are inspired from the latest achievements of Brion's PhD thesis [6]. The strategy is based on the variations of the k-space in order to detect singularities and eventually defaults. Even if the impact of the singularities might be very limited on the real part of the wave-number, it is different for the imaginary part. Indeed, this latter is much more sensitive to damping and energy dissipation that can occur when a default is present.

However, for 2D-structures, the k-space-based default detection is made difficult for two reasons: firstly, the wave-numbers depend on the wave-direction of propagation, secondly, the imaginary part is influenced by the reduction of the wave amplitude which is due to both to spatial decay and the material damping. This is the reason why, for damage location, it is necessary to use a recursive protocol based on the evaluation of the k-space variations over subdomains.

4.1 Damage index

The damage index which is used in the study is based on the differences and discrepancies between the wave-numbers. The wave-numbers are evaluated on the structure for N_f frequencies and hence compared to a reference one. The reference can be analytical or experimental. The damage index in Eqn. (5) is based on the discrepancy of either the real part and or the imaginary part of the wave-numbers.

$$Dq_g(k_{\text{measure}}, k_{\text{ref}}) = \sqrt{\frac{1}{N_f N_\theta} \sum_{n=1}^{N_f} \sum_{l=1}^{N_\theta} \left| \frac{g(k_{\text{measure}}(f_n, \theta_l)) - g(k_{\text{ref}}(f_n, \theta_l))}{g(k_{\text{ref}}(f_n, \theta_l))} \right|^2} \quad (5)$$

g is a function which will be thereafter replaced by the expression of the real part of the imaginary part of the wave-number. This index uses values of the k-space over N_f frequencies and N_θ directions.

4.2 Subdomain decomposition

The spatial field is supposed to be known over a regular spatial grid (x_n, y_m) $1 \leq n \leq N_x$, $1 \leq m \leq N_y$. Let N_w be the number of points in a subdomain in the x - or y - direction. Subdomains are hence defined like: $\Omega_{ij} = [x_i, x_i + N_w] \times [y_j, y_j + N_w]$, with $i \in [1, N_x - N_w + 1]$ and $j \in [1, N_y - N_w + 1]$. We must notice that the sub-domains can be superimposed (Fig. 3) and that N_w must be large enough in order to have a precise solution. Indeed, the sub-domains dimensions shall be large with respect to the wave-lengths.

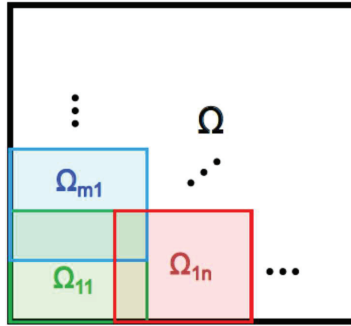


Figure 3: Sub-domain decomposition of the 2D-spatial grid [6]

From this decomposition, we can evaluate the value of the damage index for each one of these subdomains and hence retrieve the ones with the highest values. Since subdomains share locally the same areas, the location of a singularity can be assumed (Fig. 4). From this can be further confirmed by another sub-domain decomposition around only the assumed location of the singularity.

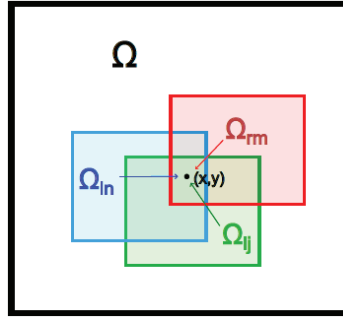


Figure 4: Damage location by damage index comparisons [6]

4.3 Validation on an isotropic plate

In order to simulate a damage on a plate, a finite element model has been designed for a plate with the following characteristics. This is an aluminum simply-supported rectangular plate $L_x = 0.95\text{m}$, $L_y = 0.6\text{m}$ with thickness $d = 0.002\text{m}$. The displacement field is calculated of a regular 44 by 55-point grid, within an inner domain of the plates surface $L_{xw} = 0.682\text{m}$ and $L_{yw} = 0.536\text{m}$.

The defect is represented by a hole close to the plates center (radius 1.2mm, 2.4mm, 6mm and 12mm). The frequencies are ranging from 50Hz to 1000Hz with 50Hz steps. From the observation of the displacement field, the damage does not seem to alter the displacement pattern as visible in Fig. 5.

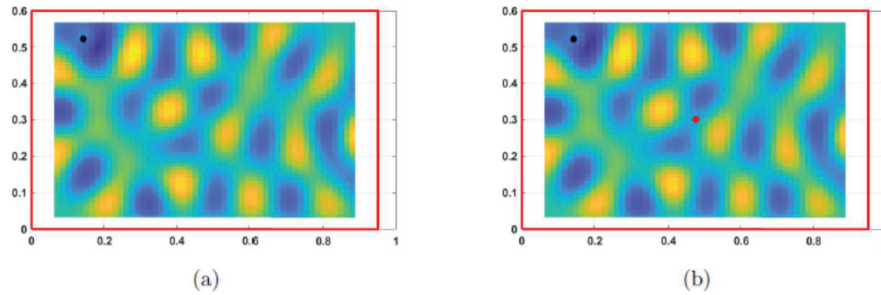


Figure 5: Displacement field at 700Hz, loading location (black dot): (a) without damage and (b) with damage (red dot) [6]

However, when evaluating the damage index over the proposed sub-domain decomposition strategy and by refining cumulating the damage index evaluation for each point of the grid, the damage index distribution is emphasizing some relevant results as visible in Fig. 6. The point load is located on the black dot and the default on the red dot.

Further examples show that the imaginary part seems to be more sensitive and more reliable

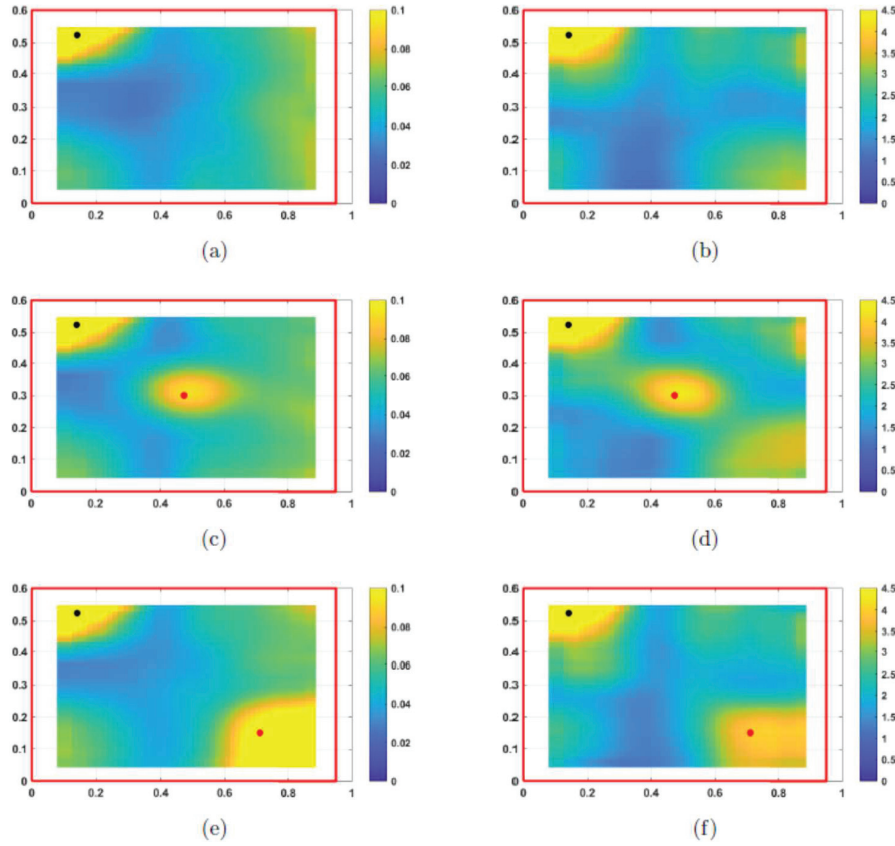


Figure 6: Surface index calculated for a 12mm hole located at red dot: (a), (c) and (e) are calculated for the real part, (b), (d) and (f) for the imaginary part; (a) and (b) are without defaults [6]

in the evaluation of the damage location, as visible in Fig. 7 and in Fig. 8. These last figure also emphasize that the AKSI k-space evaluation method seems to give a better input, as compared to the INCOME one's, to get a better contrasted damage location identification.

5 CONCLUSIONS

In this paper, a new inverse method, the Algebraic K-Space Identification 2D technique (AKSI 2D), is proposed for the extraction of the complex k-space elements from two-dimensional displacement fields. As compared to other popular 2D k-space techniques, like the IWC and the INCOME, the here-proposed method exhibits interesting performances in terms of efficiency and robustness when computing the complex k-spaces at relatively low frequencies.

This inverse method is then employed at the core of a strategy to identify the location of singularities in a plate-like structure. The strategy is based on a subdomain partition and an

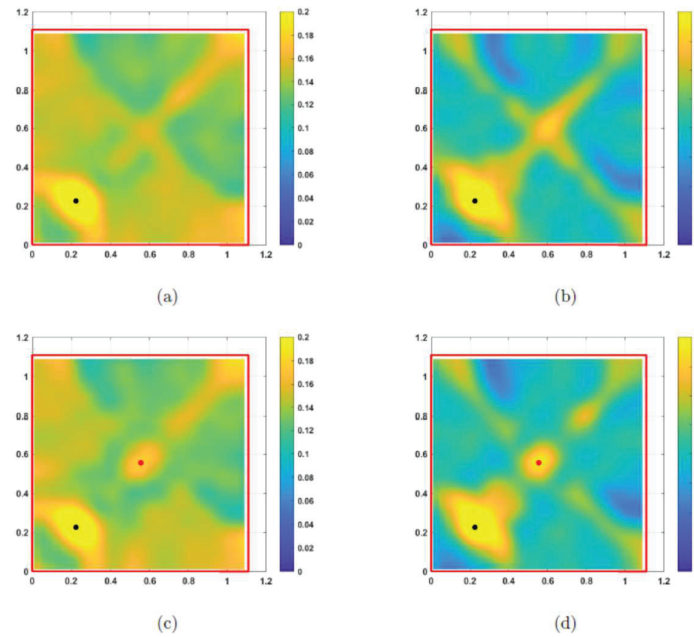


Figure 7: INCOME k-space - Surface index calculated for a 6mm hole located at black dots: (a) and (c) are calculated for the real part, (b) and (d) for the imaginary part; (a) and (b) are without defaults [6]

analysis of the local dispersion curves. A damage index location is here specifically design and give good evaluation about the damage location.

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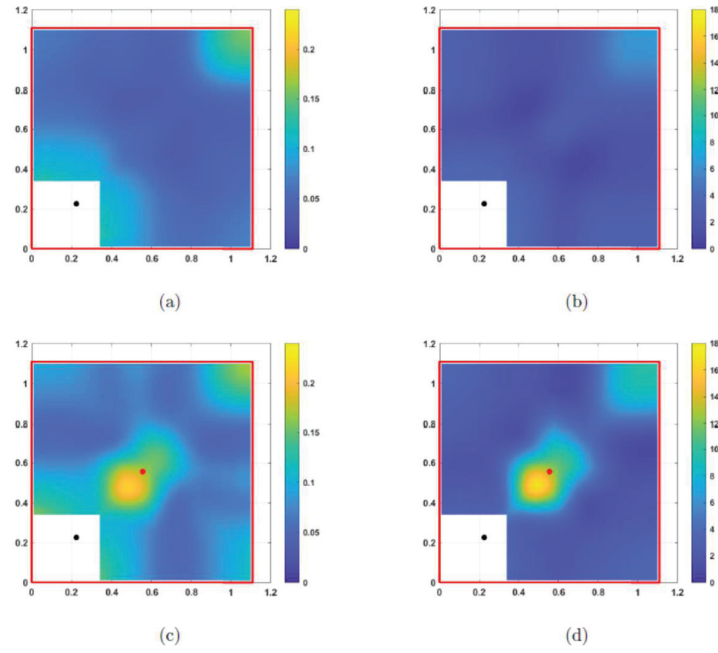


Figure 8: AKSI k-space - Surface index calculated for a 6mm hole located at black dots: (a) and (c) are calculated for the real part, (b) and (d) for the imaginary part; (a) and (b) are without defaults [6]

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