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AN APPROXIMATE PROBABILITY DENSITY FUNCTION FOR NONLINEAR VIBRATION ENERGY HARVESTING

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Abstract. *In the recent years, energy harvesting has become a promising technique as a power supply for autonomous electronic devices using ambient energy. The ambient energy can be harvested from environmental loads, such as solar radiation, wind loads, thermal gradients or from mechanical vibration in structures, vehicles, etc. In particular, energy harvesting from mechanical vibration is a promising technique because mechanical vibration is widely available in engineering environments. Furthermore, some excitations have broad bandwidth and time-varying properties. In such a case, the linear vibration energy harvester becomes inefficient. In order to improve the harvesting efficiency, a promisingly feasible way is the intentional introduction of stiffness nonlinearity into the harvester design. To model nonlinear energy harvesting, a mathematical model can be formulated using a set of coupled equations between a randomly excited spring-mass-damper system and a capacitive energy harvesting circuit using a linear electromechanical coupling coefficient. In terms of this model, the probability density function (PDF) of the response and output power is governed by the associated Fokker–Plank–Kolmogorov (FPK) equation which is hard to exactly be solved. In this paper, a solution procedure is developed to formulate an approximate joint PDF of a Duffing-type energy harvester under Gaussian white noise. The joint PDF of displacement, velocity and an electrical variable is governed by the FPK equation. First, a state-space-split method is adopted to reduce the FPK equation to the lower-dimensional FPK equation only about displacement and velocity. The stationary joint PDF of displacement and velocity can be solved exactly for a conventional Duffing system. Then, the joint PDF of displacement, velocity and the electrical variable can be approximated by the product of the obtained exact PDF and the conditional Gaussian PDF of the electrical variable. A parametric study is further conducted to show the effectiveness of the proposed solution procedure. Different nonlinearity degrees, excitation intensities and excitation means are considered in three examples. Comparison with the simulated results shows that the proposed solution procedure is effective in obtaining the joint PDF of the harvester in the examined examples, which is significant for reliability analysis on nonlinear vibration energy harvesting.*

1 INTRODUCTION

In the recent years, energy harvesting has become a promising technique as a power supply for autonomous electronic devices using ambient energy. The ambient energy can be harvested from environmental sources, such as solar radiation, wind loads, thermal gradients and mechanical vibration. In particular, energy harvesting from mechanical vibration attracts much attention because mechanical vibration is widely available in engineering environments. In the early stage, a linear vibration energy harvester is adopted and it performs well when the harvester is excited around its resonance frequency [1]. However, some excitations have broad bandwidth and time-varying properties in practice, e.g., wind loads and sea waves. In such a case, the linear vibration energy harvester becomes inefficient [2, 3]. To overcome this limitation, the techniques of broadband vibration energy harvesting have been developed. For example, multiple linear harvesters or a harvester array are proposed with individual harvesters with their resonance frequencies being comparable close to each other [4]. Active resonance tuning technique is also used to change the harvester property during the run time to match the excitation frequency [5]. This technique needs additional power supply for the active tuning [2].

Alternatively, the intentional introduction of stiffness nonlinearity into the harvester design is a feasible way to offer broadband or multiple resonant responses, which can provide a wider application in different environments [6-13]. Usually Duffing-type harvesters under Gaussian white noise have been widely considered currently [2,14]. The Duffing-type harvesters can be adopted as either a mono-stable Duffing oscillator with a hardening/softening nonlinearity [6, 9, 15] or a bi-stable Duffing oscillator with a double-well potential energy function [7]. To model the nonlinear energy harvesting, a set of coupled equations between a randomly excited spring-mass-damper system and a capacitive energy harvesting circuit using a linear electromechanical coupling coefficient [1, 2, 16]. A simple model can be established by neglecting the effective inductance of the harvesting circuit for electromagnetic inductance mechanism or the effective capacitance of the piezoelectric element for piezoelectric mechanism [9, 17, 18]. By this way, the electronic variable, such as current and voltage, can be directly formulated as the function of system velocity according to the electric equation. After that, the coupled equations are reduced to a single-degree-of-freedom equation with an effective damping coefficient. The effective damping ratio is used to account for both mechanical and electrical damping. When the single-degree-of-freedom system is modeled by a Duffing oscillator and it is excited by Gaussian white noise, the probability density function (PDF) of the response is governed by the Fokker–Plank–Kolmogorov (FPK) equation. The stationary PDF solution is well obtained [2, 14]. However, when the effective inductance or the effective capacitance is considered, the associated FPK equation is too complicated for the original coupled equations to be exactly solved [19].

In this paper, a solution procedure is developed to formulate an approximate joint PDF of a Duffing-type energy harvester under Gaussian white noise [20]. The joint PDF of displacement, velocity and an electrical variable is governed by the FPK equation. First a state-space-split method is adopted to reduce the FPK equation to the lower-dimensional FPK equation only about displacement and velocity. The stationary joint PDF of displacement and velocity can be solved exactly for a conventional Duffing system. Then the joint PDF of displacement, velocity and the electrical variable can be approximated by the product of the obtained exact PDF and the conditional Gaussian PDF of the electrical variable. A parametric study is further conducted to show the effectiveness of the proposed solution procedure. Different nonlinearity degrees, excitation intensities and excitation means are considered in three examples. Comparison with the simulated results shows that the proposed solution procedure is effective in obtaining the joint PDF of the harvester in the examined examples, which is significant for reliability analysis on nonlinear vibration energy harvesting.

2 PROBLEM FORMULATION

2.1 A Duffing-type harvesting system

A Duffing-type energy harvester under Gaussian white noise can be mathematically expressed as follows [2]

$$\begin{cases} \ddot{x} + 2\varsigma\dot{x} + (1-r)x + \delta x^3 + \kappa^2 z = W(t) \\ \dot{z} + \alpha z = \dot{x} \end{cases} \tag{1}$$

where $\ddot{x}$, $\dot{x}$ and x are acceleration, velocity and displacement, respectively; z is an electrical variable, e.g., current or voltage; ς is a mechanical damping ration; r is a tuning parameter; δ denotes the coefficient of cubic nonlinearity; κ is a linear dimensionless electromechanical coupling coefficient; α is the ratio between the mechanical and electrical time constants of the harvester. The above equation is formulated in a non-dimensional form which can be applicable to all similar devices. $W(t)$ is Gaussian white noise with a mean as is given below

$$W(t) = m + W_0(t) \tag{2}$$

and its expectation is

$$E[W(t)] = m\,,\ \ E[W_0(t)W_0(t+\tau)] = 2\pi K\delta(\tau) \tag{3}$$

where m is the mean of Gaussian white noise; $W_0(t)$ is a zero mean Gaussian white noise.

2.2 Equivalent linearization method

First a standard equivalent linearization method is used to obtain the approximate mean and variance of the original system. Using the obtained mean and variance, a solution procedure is proposed to formulate the joint PDF solution of the response and electrical variable.

Considering Eqs. (1) and (2), a linear equivalent system can be expressed as follows

$$\begin{cases} \ddot{x} + 2\varsigma\dot{x} + k_e x + \kappa^2 z = m + W_0(t) \\ \dot{z} + \alpha z = \dot{x} \end{cases} \tag{4}$$

Letting $x = x_1$, $\dot{x} = x_2$ and $z = x_3$, Eq. (4) can be expressed by a set of first-order differential equations

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -2\zeta x_2 - k_e x_1 - \kappa^2 x_3 + m + W_0(t) \\ \dot{x}_3 = -\alpha x_3 + x_2 \end{cases} \tag{5}$$

Using the standard equivalent linearization method, the following results are obtained

$$k_e = (1-r) + \delta E[x_1^4] \big/ E[x_1^2] \tag{6}$$

$$E[x_1] = \mu_{x_1} = \frac{m}{k_e}\,,\ \ E[x_2] = \mu_{x_2} = 0\,,\ \ E[x_3] = \mu_{x_3} = 0 \tag{7}$$

$$E[x_1^2] = \frac{\pi K}{k_e} \frac{k_e + \alpha^2 + 2\alpha\zeta}{2\zeta(k_e + \alpha^2 + 2\alpha\zeta) + \kappa^2(\alpha + 2\zeta)} + \frac{m^2}{k_e^2} \tag{8}$$

$$E[x_2^2] = \pi K \frac{k_e + \alpha^2 + 2\alpha\zeta + \kappa^2}{2\zeta(k_e + \alpha^2 + 2\alpha\zeta) + \kappa^2(\alpha + 2\zeta)} \tag{9}$$

$$E[x_3^2] = \pi K \frac{1}{2\zeta(k_e + \alpha^2 + 2\alpha\zeta) + \kappa^2(\alpha + 2\zeta)} \tag{10}$$

$$E[x_1 x_2] = 0 \tag{11}$$

$$E[x_1 x_3] = E[x_3^2] \tag{12}$$

$$E[x_2 x_3] = \alpha E[x_3^2] \tag{13}$$

Simultaneously solving Eqs. (6) and (8), k_e and $E[x_1^2]$ are obtained. Subsequently, other equations can be solved. It is well known the result of a standard equivalent linearization method is Gaussian. The Gaussian PDF solution $p_G(x_1, x_2, x_3)$ of the response and electrical variable is given below

$$p_G(x_1, x_2, x_3) = \frac{\exp\left(-\chi^2 / 2\Sigma^2\right)}{2\sqrt{2}\pi^{3/2}\sqrt{1 - (\rho_{12}^2 + \rho_{13}^2 + \rho_{23}^2) + 2\rho_{12}\rho_{13}\rho_{23}}} \tag{14}$$

$$\begin{aligned} \chi^2 = & \frac{(x_1 - \mu_{x_1})^2}{\sigma_1^2}(\rho_{23}^2 - 1) + \frac{x_2^2}{\sigma_2^2}(\rho_{13}^2 - 1) + \frac{x_3^2}{\sigma_3^2}(\rho_{12}^2 - 1) \\ & + 2\left[\frac{(x_1 - \mu_{x_1})}{\sigma_1}\frac{x_2}{\sigma_2}(\rho_{12} - \rho_{13}\rho_{23}) + \frac{(x_1 - \mu_{x_1})}{\sigma_1}\frac{x_3}{\sigma_3}(\rho_{13} - \rho_{12}\rho_{23}) + \frac{x_2}{\sigma_2}\frac{x_3}{\sigma_3}(\rho_{23} - \rho_{12}\rho_{13})\right] \end{aligned} \tag{15}$$

$$\Sigma^2 = \rho_{12}^2 + \rho_{13}^2 + \rho_{23}^2 - 2\rho_{12}\rho_{13}\rho_{23} - 1 \tag{16}$$

$$\sigma_i = \sqrt{E[(x_i - \mu_{x_i})^2]} \tag{17}$$

$$\rho_{ij} = \frac{E[(x_i - \mu_{x_i})(x_j - \mu_{x_j})]}{\sigma_i \sigma_j} \tag{18}$$

The conditional Gaussian PDF $\bar{q}(x_3 | x_1, x_2)$ is still Gaussian, its mean and variance are

$$\mu(x_3 | x_1, x_2) = \mu_{x_3} + \rho_{13}\frac{\sigma_3}{\sigma_1}(x_1 - \mu_{x_1}) + \rho_{23}\frac{\sigma_3}{\sigma_2}(x_2 - \mu_{x_2}) \tag{19}$$

$$\sigma^2(x_3 | x_1, x_2) = (1 - \rho_{13}^2 - \rho_{23}^2)\sigma_3^2 \tag{20}$$

In the stationary case, μ_{x_2} and μ_{x_3} are all zero. Therefore,

$$\mu(x_3 | x_1, x_2) = \int_{-\infty}^{+\infty} x_3 \bar{q}(x_3 | x_1, x_2) dx_3 = \rho_{13}\frac{\sigma_3}{\sigma_1}(x_1 - \mu_{x_1}) + \rho_{23}\frac{\sigma_3}{\sigma_2}x_2 \tag{21}$$

2.3 State-space-split method

Similarly, Eq. (1) can be also expressed in a form of first-order differential equations including Eq. (2)

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -2\zeta x_2 - (1-r)x_1 - \delta x_1^3 - \kappa^2 x_3 + m + W_0(t) \\ \dot{x}_3 = -\alpha x_3 + x_2 \end{cases} \tag{22}$$

The joint PDF solution $p(x_1, x_2, x_3, t)$ of Eq. (22) is governed by the FPK equation

$$\frac{\partial p}{\partial t} = -x_2 \frac{\partial p}{\partial x_1} + \frac{\partial}{\partial x_2}\left\{\left[2\zeta x_2 + (1-r)x_1 + \delta x_1^3 + \kappa^2 x_3 - m\right]p\right\} + \frac{\partial}{\partial x_3}\left\{(\alpha x_3 - x_2)p\right\} + \pi K \frac{\partial^2 p}{\partial x_2^2} \tag{23}$$

Herein, the stationary PDF $p(x_1, x_2, x_3)$ is considered. Eq. (23) is reduced to the stationary case

$$-x_2 \frac{\partial p}{\partial x_1} + \frac{\partial}{\partial x_2}\left\{\left[2\zeta x_2 + (1-r)x_1 + \delta x_1^3 + \kappa^2 x_3 - m\right]p\right\} + \frac{\partial}{\partial x_3}\left\{(\alpha x_3 - x_2)p\right\} + \pi K \frac{\partial^2 p}{\partial x_2^2} = 0 \tag{24}$$

Integrating Eq. (24) with respect to x_3 results in

$$\begin{aligned} &-x_2 \frac{\partial}{\partial x_1}\int_{R^{x_3}} p dx_3 + \frac{\partial}{\partial x_2}\left\{\left[2\zeta x_2 + (1-r)x_1 + \delta x_1^3 - m\right]\int_{R^{x_3}} p dx_3 + \kappa^2 \int_{R^{x_3}} x_3 p dx_3\right\} \\ &+ \int_{R^{x_3}} \frac{\partial}{\partial x_3}\left\{(\alpha x_3 - x_2)p\right\}dx_3 + \pi K \frac{\partial^2}{\partial x_2^2}\int_{R^{x_3}} p dx_3 = 0 \end{aligned} \tag{25}$$

Some simplification and assumptions are made on Eq. (25). First

$$\int_{R^{x_3}} p dx_3 = \int_{-\infty}^{+\infty} p(x_1, x_2, x_3) dx_3 = p(x_1, x_2) \tag{26}$$

Second, the probability and probability flux is assumed to be zero at infinite boundary, i.e.,

$$\begin{cases} \lim\limits_{x_i \to \pm\infty} p(x_1, x_2, x_3) = 0, \quad i = 1,2,3 \\ \lim\limits_{x_i \to \pm\infty} \left\{(\alpha x_3 - x_2)p(x_1, x_2, x_3)\right\} = 0, \quad i = 1,2,3 \end{cases} \tag{27}$$

Consequently, the third term in Eq. (25)

$$\int_{R^{x_3}} \frac{\partial}{\partial x_3}\left\{(\alpha x_3 - x_2)p\right\}dx_3 = \int_{-\infty}^{+\infty} \frac{\partial}{\partial x_3}\left\{(\alpha x_3 - x_2)p\right\}dx_3 = 0 \tag{28}$$

Third, in Eq. (25)

$$\int_{R^{x_3}} x_3 p dx_3 = \int_{R^{x_3}} x_3 p(x_1, x_2, x_3) dx_3 = \int_{-\infty}^{+\infty} x_3 q(x_3 | x_1, x_2) p(x_1, x_2) dx_3 = p(x_1, x_2)\int_{-\infty}^{+\infty} x_3 q(x_3 | x_1, x_2) dx_3 \tag{29}$$

where $q(x_3 | x_1, x_2)$ is the conditional PDF of x_3 given with x_1 and x_2. Therefore, Eq. (25) is further reformulated as

$$-x_2 \frac{\partial p}{\partial x_1} + \frac{\partial}{\partial x_2}\left\{\left[2\zeta x_2 + (1-r)x_1 + \delta x_1^3 - m + \kappa^2 \int_{-\infty}^{+\infty} x_3 q(x_3 | x_1, x_2) dx_3\right]p\right\} + \pi K \frac{\partial^2 p}{\partial x_2^2} = 0 \tag{30}$$

In Eq. (30), $q(x_3|x_1,x_2)$ is unknown but it can be approximated by $\bar{q}(x_3|x_1,x_2)$. Considering Eq. (21),

$$\begin{aligned}0 &= -x_2\frac{\partial\tilde{p}}{\partial x_1}+\frac{\partial}{\partial x_2}\left\{\left[2\zeta x_2+(1-r)x_1+\delta x_1^3-m+\kappa^2\int_{-\infty}^{+\infty}x_3\bar{q}(x_3|x_1,x_2)dx_3\right]\tilde{p}\right\}+\pi K\frac{\partial^2\tilde{p}}{\partial x_2^2}\\ &= -x_2\frac{\partial\tilde{p}}{\partial x_1}+\frac{\partial}{\partial x_2}\left\{\left[2\zeta x_2+(1-r)x_1+\delta x_1^3-m+\kappa^2\rho_{13}\frac{\sigma_3}{\sigma_1}x_1-\kappa^2\rho_{13}\frac{\sigma_3}{\sigma_1}\mu_{x_1}+\kappa^2\rho_{23}\frac{\sigma_3}{\sigma_2}x_2\right]\tilde{p}\right\}+\pi K\frac{\partial^2\tilde{p}}{\partial x_2^2} \quad (31)\\ &= -x_2\frac{\partial\tilde{p}}{\partial x_1}+\frac{\partial}{\partial x_2}\left\{\left[(2\zeta+\kappa^2\rho_{23}\frac{\sigma_3}{\sigma_2})x_2+(1-r+\kappa^2\rho_{13}\frac{\sigma_3}{\sigma_1})x_1+\delta x_1^3-m-\kappa^2\rho_{13}\frac{\sigma_3}{\sigma_1}\mu_{x_1}\right]\tilde{p}\right\}+\pi K\frac{\partial^2\tilde{p}}{\partial x_2^2}\end{aligned}$$

where $\tilde{p}$ represents $\tilde{p}(x_1,x_2)$ which is an approximation to $p(x_1,x_2)$.

After that, Eq. (24) about x_1, x_2 and x_3 is reduced to Eq. (31) only about x_1 and x_2. Meanwhile, the exact stationary PDF solution to Eq. (31) is known as [2,14]

$$\tilde{p}(x_1,x_2)=C\exp\left\{-\frac{(2\zeta+\kappa^2\rho_{23}\frac{\sigma_3}{\sigma_2})}{2\pi K}\left[x_2^2+(1-r+\kappa^2\rho_{13}\frac{\sigma_3}{\sigma_1})x_1^2+0.5\delta x_1^4-2mx_1-2\kappa^2\rho_{13}\frac{\sigma_3}{\sigma_1}\mu_{x_1}x_1\right]\right\} \quad (32)$$

where C is a normalized constant. The joint PDF of x_1, x_2 and x_3 can be approximated by the product of the obtained exact PDF and the conditional Gaussian PDF of the electrical variable as follows

$$\begin{aligned}\tilde{p}(x_1,x_2,x_3) &= \bar{q}(x_3|x_1,x_2)\tilde{p}(x_1,x_2)\\ &= \frac{1}{\sqrt{2\pi}\sigma(x_3|x_1,x_2)}\exp\left\{-\frac{[x_3-\mu(x_3|x_1,x_2)]^2}{2\sigma^2(x_3|x_1,x_2)}\right\}\tilde{p}(x_1,x_2)\\ &= A\exp\left\{-\frac{[x_3-\rho_{13}\frac{\sigma_3}{\sigma_1}(x_1-\mu_{x_1})+\rho_{23}\frac{\sigma_3}{\sigma_2}x_2]^2}{2(1-\rho_{13}^2-\rho_{23}^2)\sigma_3^2}\right.\\ &\quad\left.-\frac{(2\zeta+\kappa^2\rho_{23}\frac{\sigma_3}{\sigma_2})}{2\pi K}\left[x_2^2+(1-r+\kappa^2\rho_{13}\frac{\sigma_3}{\sigma_1})x_1^2+0.5\delta x_1^4-2mx_1-2\kappa^2\rho_{13}\frac{\sigma_3}{\sigma_1}\mu_{x_1}x_1\right]\right\}\end{aligned} \quad (33)$$

where A is also a normalized constant and $A=C\big/\sqrt{2\pi(1-\rho_{13}^2-\rho_{23}^2)}\sigma_3$. Therefore,

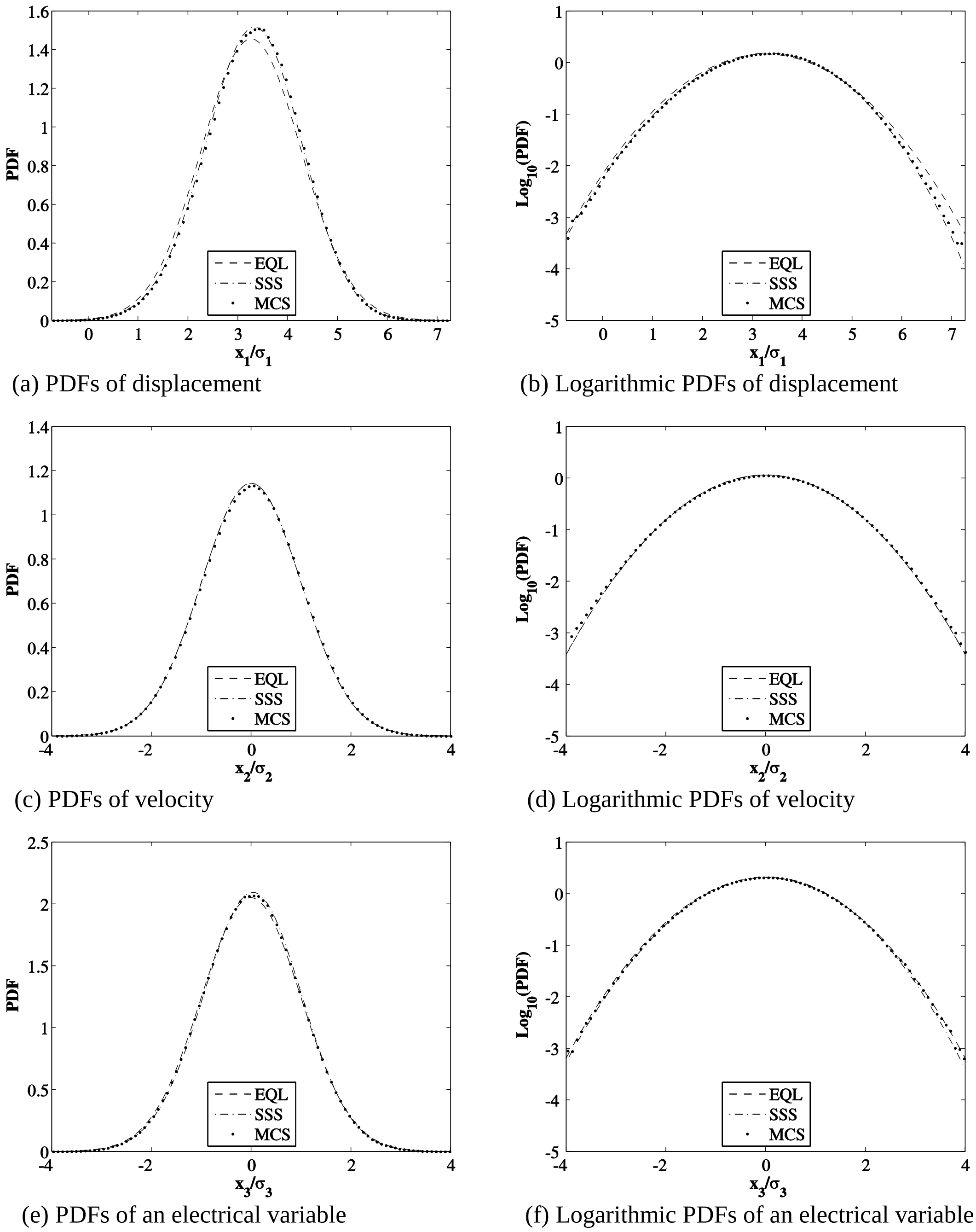
$$\tilde{p}(x_1)=\int_{-\infty}^{+\infty}\tilde{p}(x_1,x_2)dx_2\ ,\ \tilde{p}(x_2)=\int_{-\infty}^{+\infty}\tilde{p}(x_1,x_2)dx_1\ ,\ \tilde{p}(x_3)=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\tilde{p}(x_1,x_2,x_3)dx_1dx_2 \quad (34)$$

3 NUMERICAL ANALYSIS

Herein, different nonlinearity degrees, excitation intensities and excitation means are considered in three examples to show the effectiveness of the proposed solution procedure. Monte Carlo simulation is also conducted with a sample size is 2×10^7 to obtain the stationary response. The results of the standard equivalent linearization, proposed solution procedure and simulation are denoted as EQL, SSS and MCS, respectively. The values of the PDF and its logarithmic values for displacement, velocity and the electrical variable are presented respectively in each figure to show the global and tail behaviors.

3.1 Weak nonlinearity in displacement

The first example is about the case of weak nonlinearity in displacement. According to Eqs. (1) through (3), the parameters are given as follows: $\varsigma = 0.05$, $r = 0$, $\delta = 0.1$, $\kappa^2 = 1$, $\alpha = 1$, $m = 1$, and $2\pi K = 0.1$. Because EQL denotes a Gaussian PDF distribution, the PDF distribution of each variable are nearly Gaussian as shown in Fig. 1 except that the PDF of displacement has a nonzero mean and it also shows a little non-symmetric distribution. This is because in the weak nonlinearity and low-level excitation intensity, the nonlinear system behaves very closely as a linear system does. It is well known, the response of a linear system under Gaussian white noise is still Gaussian. Therefore, the nearly Gaussian behaviors are formulated.

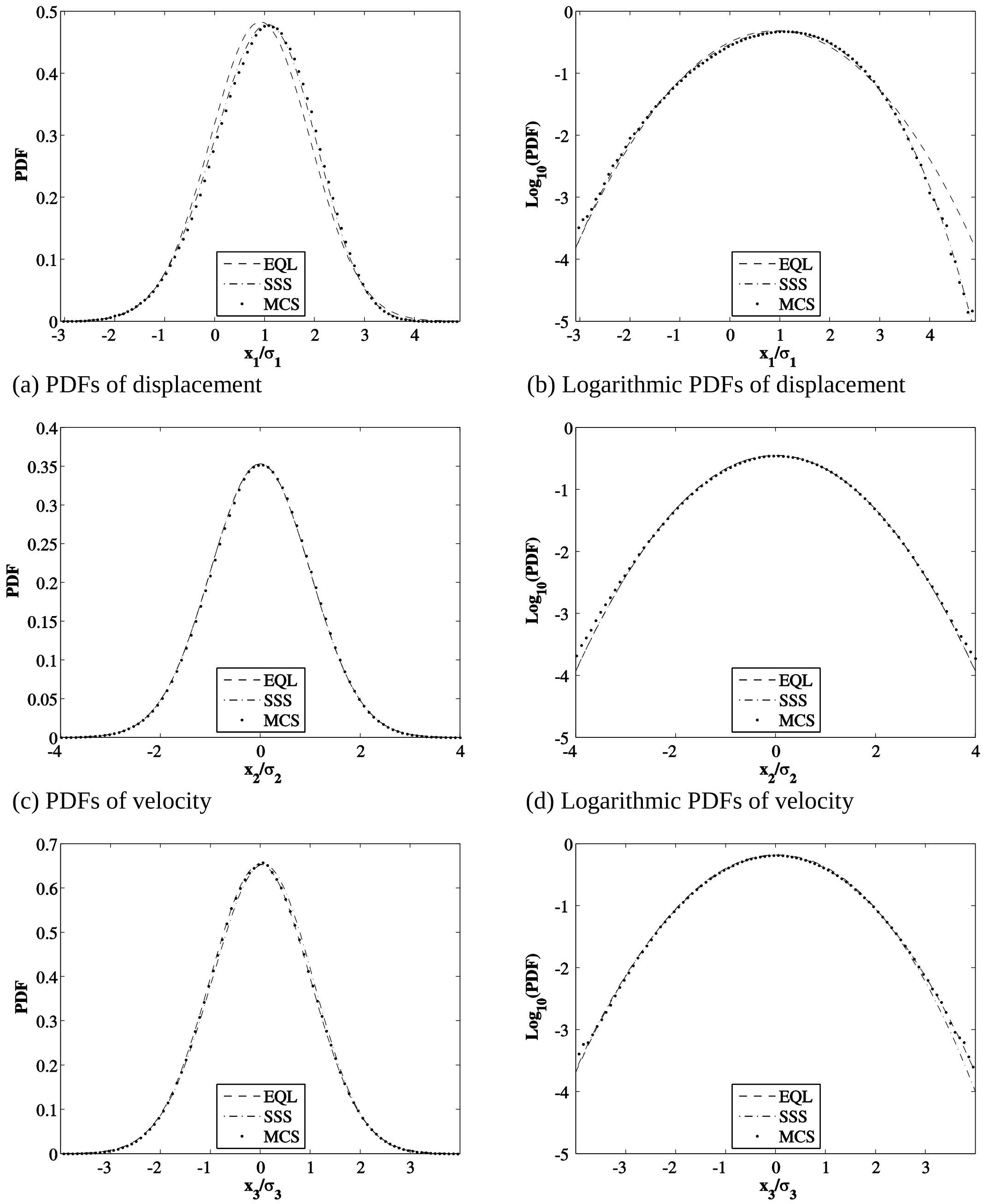


(a) PDFs of displacement
(b) Logarithmic PDFs of displacement
(c) PDFs of velocity
(d) Logarithmic PDFs of velocity
(e) PDFs of an electrical variable
(f) Logarithmic PDFs of an electrical variable

Figure 1: Comparison of PDFs for example 1.

3.2 High-level excitation intensity

The second example is about the case of high-level excitation intensity. According to Eqs. (1) through (3), the parameters are given as follows: $\varsigma = 0.05$, $r = 0$, $\delta = 0.1$, $\kappa^2 = 1$, $\alpha = 1$, $m = 1$, and $2\pi K = 1$. Figs. 2(a) and 2(b) shows good agreement between SSS and MCS and the PDF of displacement has a nonzero mean and non-symmetric distribution. By contrast, the PDF of velocity is nearly Gaussian as shown in Figs. 2(c) and 2(d). The similar behavior is also observed in the case of the electrical variable (Figs. 2(e) and 2(f)). Therefore, the PDF distribution of velocity and the electrical variable is less affected by the one of displacement in this examined example.



(a) PDFs of displacement (b) Logarithmic PDFs of displacement

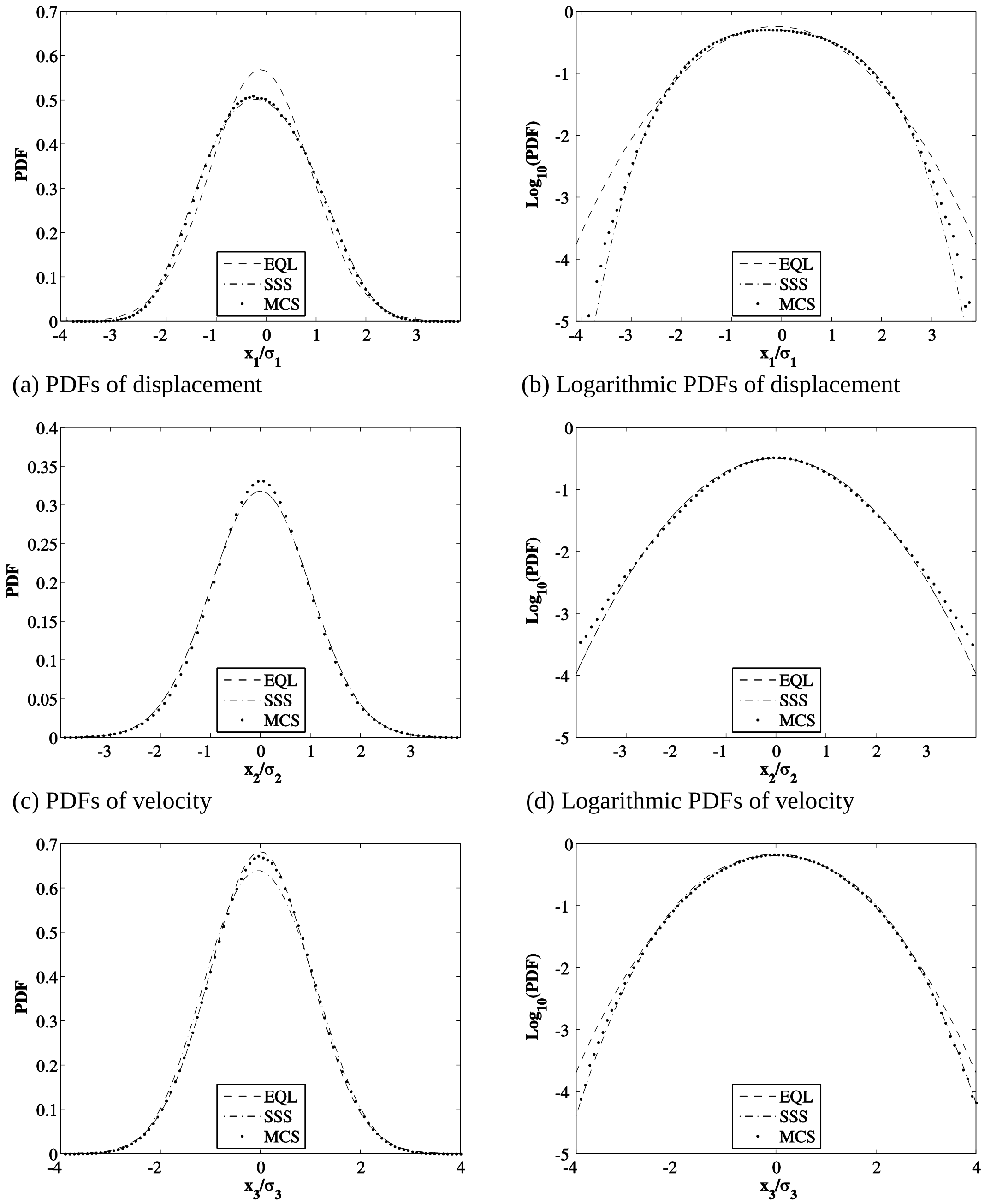
(c) PDFs of velocity (d) Logarithmic PDFs of velocity

(e) PDFs of an electrical variable (f) Logarithmic PDFs of an electrical variable

Figure 2: Comparison of PDFs for example 2.

3.3 Strong nonlinearity in displacement

The last example is about the case of strong nonlinearity in displacement. According to Eqs. (1) through (3), the parameters are given as follows: $\varsigma = 0.05$, $r = 0$, $\delta = 1$, $\kappa^2 = 1$, $\alpha = 1$, $m = -0.2$, and $2\pi K = 1$. As shown in Figs. 3(a) and 3(b), MCS significantly differs from EQL indicating that the PDF distribution of displacement is highly non-Gaussian due to the presence of strong nonlinearity. In such a case, the PDF of velocity is still nearly Gaussian although the tail differs a little from EQL in Fig. 3(d). For the electrical variable, its PDF distribution becomes non-Gaussian in the tail region as shown in Fig. 3(f). Therefore, the strong nonlinearity in displacement also affects the ones of velocity and the electrical variable.



(a) PDFs of displacement

(b) Logarithmic PDFs of displacement

(c) PDFs of velocity

(d) Logarithmic PDFs of velocity

(e) PDFs of an electrical variable

(f) Logarithmic PDFs of an electrical variable

Figure 3: Comparison of PDFs for example 3.

4 CONCLUSIONS

This paper develops a solution procedure to formulate an approximate joint PDF solution of a Duffing-type energy harvester under Gaussian white noise. The joint PDF of displacement, velocity and an electrical variable is governed by the FPK equation. A state-space-split method is used to reduce the FPK equation to the lower-dimensional FPK equation only about displacement and velocity. The stationary joint PDF of displacement and velocity can be solved exactly. After that, the joint PDF of displacement, velocity and the electrical variable is formulated as the product of the obtained exact PDF and the conditional Gaussian PDF of the electrical variable. Three examples are considered in the following numerical analysis. Different nonlinearity degrees, excitation intensities and excitation means are adopted to show the effectiveness of the proposed solution procedure. Comparison with the simulated results shows that the proposed solution procedure is effective in obtaining the joint PDF of the harvester in the examined examples, which is significant for reliability analysis on nonlinear vibration energy harvesting. The comparison further shows that the PDF of displacement has a nonzero mean and a non-symmetrical distribution, which differs significantly from a Gaussian distribution. In contrast, the PDFs of velocity and the electrical variable are nearly Gaussian, which is less affected by nonzero mean excitation and nonlinearity in displacement. However, when strong nonlinearity and high-level excitation intensity exists, the PDFs of velocity and the electrical variable also become a little non-Gaussian.

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