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SOLUTION OF GENERALIZED DENSITY EVOLUTION EQUATION VIA A PHASE SPACE RECONSTRUCTION METHOD

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Abstract. *In stochastic dynamic response analysis, getting the probability density function and its evolution process is one of the most important problem. A variety of theoretical and numerical methods have made progress on linear stochastic system. However, the analysis of non-linear stochastic dynamic systems is still a challenging problem despite great efforts have been many in this area. As a newly developed method, the probability density evolution method (PDEM) is capable of capturing the instantaneous probability density function (PDF) of stochastic dynamic responses of systems. In this paper, based on the formal solution of the generalized density evolution equation (GDEE), a new method named "phase space reconstruction method" (PSRM) is introduced. In addition, it is found that the instantaneous PDF of dynamic responses of highly non-linear systems can be obtained precisely and effectively. At last, some examples, including a Riccati oscillator, a SDOF oscillator, and a Duffing oscillator, are studied. The results shows the effectiveness and preciseness of PSRM in getting the PDF of dynamic response.*

1 INTRODUCTION

With the rapid development of computational and theoretical means, it is widely recognized that stochastic methodologies are indispensable. The factors in engineering practice, such as the material properties, the external loads and the boundary conditions are inherently uncontrolled. As a consequence, stochastic dynamics gained increasing interest and has been extensively studied in the past decades.

In engineering area, the most widely used three available methods in stochastic dynamic response analysis are the Monte Carlo simulation method [1] , the random perturbation technique [2] and the orthogonal polynomials expansion method [3]. However, the existing methods may have difficulties to obtain stochastic dynamic responses of structures involving high nonlinearities with tradeoff of efficiency and accuracy. In addition, all the approaches mentioned above focus on obtaining moment information such as the mean and the standard deviation rather than the probability density function (PDF) which could completely capture the time-varying property of the structures. In recent years, a newly developed method named probability density evolution method (PDEM) proposed by Li and Chen [4, 5] is capable of capturing the instantaneous probability density function (PDF) of dynamic responses of structures. Some numerical algorithms such as the finite difference method and representative point sets strategy are introduced to obtain the numerical solution of generalized density evolution equation (GDEE).

In this paper, a more accurate, efficient method named the "phase space reconstruction method" (PSRM) is presented for the solution of GDEE. Three typical SDOF oscillators with high nonlinearity are studied and the probabilistic solutions of PSRM demonstrate the accuracy, efficiency and convenience of this method.

2 THE GENERALIZED DENSITY EVOLUTION EQUATION

Without loss of generality, a stochastic dynamical system can be expressed as

$$\dot{\mathbf{X}}(t) = \mathbf{\Phi}(\mathbf{X}, \mathbf{\Theta}, t) \tag{1}$$

with the initial condition

$$\mathbf{X}(t)\big|_{t=0} = \mathbf{x}_0 \tag{2}$$

where $\mathbf{X} = (X_1, X_2, \cdots, X_N)^{\mathrm{T}}$ is the state vector consisting of N components $X_i(t)$, $\mathbf{\Phi} = (\mathit{\Phi}_1, \mathit{\Phi}_2, \cdots, \mathit{\Phi}_N)^{\mathrm{T}}$ the dynamic operator, $\mathbf{\Theta} = (\mathit{\Theta}_1, \mathit{\Theta}_2, \cdots, \mathit{\Theta}_s)^{\mathrm{T}}$ the s-dimensional vector with the joint probability density function (PDF) $p_{\mathbf{\Theta}}(\boldsymbol{\theta})$, $\mathbf{x}_0 = (x_{0,1}, x_{0,2}, \cdots, x_{0,s})^{\mathrm{T}}$ the initial value vector with the PDF $p_{\mathbf{X}_0}(\mathbf{x})$.

It is well known that, under certain regularity conditions, the solution of Eq. (1) and Eq. (2) exists and is unique. This solution is expressible in the form

$$\mathbf{X}(t) = \mathbf{H}(\mathbf{\Theta}, t) \quad \text{or} \quad X_j(t) = H_j(\mathbf{\Theta}, t), \quad j = 1, 2, \cdots, N \tag{3}$$

Then, according to the principle of preservation of probability, there exists the generalized density evolution equation [5]

$$\frac{\partial p_{\mathbf{X\Theta}}(\mathbf{x}, \boldsymbol{\theta}, t)}{\partial t} + \sum_{j=1}^{N} \dot{X}_j(\boldsymbol{\theta}, t) \cdot \frac{\partial p_{\mathbf{X\Theta}}(\mathbf{x}, \boldsymbol{\theta}, t)}{\partial x_j} = 0 \tag{4}$$

with the initial condition

$$p_{X\Theta}(\mathbf{x},\boldsymbol{\theta},t)=\delta(\mathbf{x}-\mathbf{x}_0)\,p_{\Theta}(\boldsymbol{\theta}) \tag{5}$$

As a partial differential equation, the analytical solution of Eq. (4) is hard to achieve, but its numerical solution is usually available [5].

3 PHASE SPACE RECONSTRUCTION METHOD

It is noticeable that the generalized density evolution equation (Eq.(4) and (5)) is a first order quasi-linear partial differential equation. The method of characteristics thus can be used to obtain the analytical solution. Actually, the formal solution of Eq. (4) and Eq. (5) is

$$p_{X\Theta}(\mathbf{x},\boldsymbol{\theta},t)=\delta(\mathbf{x}-\mathbf{H}(\boldsymbol{\theta},t))\,p_{\Theta}(\boldsymbol{\theta}) \tag{6}$$

Let

$$\mathbf{G}(\boldsymbol{\theta},\mathbf{x},t)=\mathbf{x}-\mathbf{H}(\boldsymbol{\theta},t) \tag{7}$$

then Eq. (6) becomes

$$p_{X\Theta}(\mathbf{x},\boldsymbol{\theta},t)=\delta[\mathbf{G}(\boldsymbol{\theta},\mathbf{x},t)]\,p_{\Theta}(\boldsymbol{\theta}) \tag{8}$$

For $a\neq 0$, there is

$$\delta(a\mathbf{x})=\frac{1}{|a|}\,\delta(\mathbf{x}) \tag{9}$$

More generally, the delta function of $g(\mathbf{x})$ is given by [6]

$$\delta[g(\mathbf{x})]=\sum_i \frac{\delta(\mathbf{x}-\mathbf{x}_i)}{|g'(\mathbf{x}_i)|} \tag{10}$$

where $\mathbf{x}_i$ are the roots of $g(\mathbf{x})$. Substituting Eq. (10) in the formal solution of the PDEE (8) leads to

$$\delta[\mathbf{G}(\boldsymbol{\theta},\mathbf{x},t)]=\sum_{i=1}^{N_{\mathrm{sol}}}\frac{\delta(\boldsymbol{\theta}-\tilde{\boldsymbol{\theta}}_i(\mathbf{x},t))}{\left|\left.\dfrac{\partial \mathbf{G}(\boldsymbol{\theta},\mathbf{x},t)}{\partial\boldsymbol{\theta}}\right|_{\boldsymbol{\theta}=\tilde{\boldsymbol{\theta}}_i(\mathbf{x},t)}\right|} \tag{11}$$

where $\tilde{\boldsymbol{\theta}}_i(\mathbf{x},t)$ are the root lines of $\mathbf{G}(\boldsymbol{\theta},\mathbf{x},t)=0$, N_{sol} is the number of the root lines of $\mathbf{G}(\boldsymbol{\theta},\mathbf{x},t)=0$.

The Dirac delta function has the fundamental property that

$$\int_{-\infty}^{\infty} f(\mathbf{x})\delta(\mathbf{x}-a)\mathrm{d}\mathbf{x}=f(a) \tag{12}$$

Combing Eqs. (8), (11), and (12) and taking the integral of $\boldsymbol{\theta}$ will obtain

$$\begin{aligned}p_{X}(\mathbf{x},t)&=\int_{\Omega_\Theta} p_{X\Theta}(\mathbf{x},\boldsymbol{\theta},t)\mathrm{d}\boldsymbol{\theta}=\int_{\Omega_\Theta}\delta[\mathbf{G}(\boldsymbol{\theta},\mathbf{x},t)]\,p_{\Theta}(\boldsymbol{\theta})\mathrm{d}\boldsymbol{\theta}\\ &=\int_{\Omega_\Theta}\sum_{i=1}^{N_{\mathrm{sol}}}\frac{\delta(\boldsymbol{\theta}-\tilde{\boldsymbol{\theta}}_i(\mathbf{x},t))}{\left|\left.\dfrac{\partial \mathbf{G}(\boldsymbol{\theta},\mathbf{x},t)}{\partial\boldsymbol{\theta}}\right|_{\boldsymbol{\theta}=\tilde{\boldsymbol{\theta}}_i(\mathbf{x},t)}\right|}\,p_{\Theta}(\boldsymbol{\theta})\mathrm{d}\boldsymbol{\theta}=\sum_{i=1}^{N_{\mathrm{sol}}}\frac{p_{\Theta}(\tilde{\boldsymbol{\theta}}_i(\mathbf{x},t))}{\left|\left.\dfrac{\partial \mathbf{G}(\boldsymbol{\theta},\mathbf{x},t)}{\partial\boldsymbol{\theta}}\right|_{\boldsymbol{\theta}=\tilde{\boldsymbol{\theta}}_i(\mathbf{x},t)}\right|}H[\Omega_{\Theta_i}]\end{aligned} \tag{13}$$

where $H(\cdot)$ is Heaviside's function, Ω_{Θ_i} is the regions of integration.

4 PHASE SPACE RECONSTRUCTION METHOD

Three typical oscillators with high nonlinearity are studied in this section to verify the applicability of the proposed method.

4.1 A Riccati oscillator

The equation of this oscillator reads

$$\dot{X} + \theta \cdot X^2 - X = 0 \tag{14}$$

with the initial condition $X(0) = 1$. The random variable θ follows the standard normal distribution with PDF

$$p_\Theta(\theta) = \frac{1}{\sqrt{2\pi}} \exp(-\frac{\theta^2}{2}) \tag{15}$$

The analytical solution of Eq. (14) is

$$X(\theta, t) = \frac{\exp(t)}{1 + \theta[\exp(t) - 1]} \tag{16}$$

Let

$$G(\theta, x, t) = x - X(\theta, t) = x - \frac{\exp(t)}{1 + \theta[\exp(t) - 1]} \tag{17}$$

When $t = 1$, there is

$$G(\theta, x) = x - X(\theta) = x - \frac{\exp(1)}{1 + \theta[\exp(1) - 1]} \tag{18}$$

The curve of $G(\theta, x) = 0$ when $t = 1$ are shown in Figures 1.

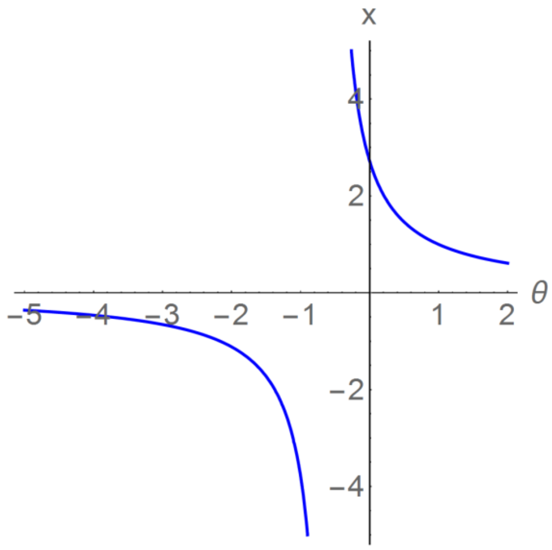

Figure 1: Page layout.

According to Eq. (17) and Figure 1, we have

$$\delta[G(\theta,x)] = \frac{\delta(\theta-\tilde{\theta}_1(x))}{\left|\frac{\partial G(\theta,x)}{\partial\theta}\Big|_{\theta=\tilde{\theta}_1(x)}\right|} = \frac{\delta(\theta-\frac{\exp(1)-x}{x[\exp(1)-1]})}{\left|\frac{\exp(1)[\exp(1)-1]}{\{1+\theta[\exp(1)-1]\}^2}\Big|_{\theta=\tilde{\theta}_1(x)}\right|} = \frac{\delta(\theta-\frac{\exp(1)-x}{x[\exp(1)-1]})}{\left|\frac{\exp(1)[\exp(1)-1]}{\left\{1+\frac{\exp(1)-x}{x[\exp(1)-1]}[\exp(1)-1]\right\}^2}\right|} \tag{19}$$

Then Eq. (13) can be rewritten in the form

$$p_X(x,t=1) = p_{X_1}(x) = \frac{p_\Theta(\tilde{\theta}_1(x))}{\left|\frac{\partial G(\theta,x)}{\partial\theta}\Big|_{\theta=\tilde{\theta}_1(x)}\right|}H[\Omega_{\Theta_1}]$$

$$= \frac{p_\Theta(\frac{\exp(1)-x}{x[\exp(1)-1]})}{\left\|\frac{\exp(1)[\exp(1)-1]}{\left\{1+\frac{\exp(1)-x}{x[\exp(1)-1]}[\exp(1)-1]\right\}^2}\right\|}\cdot H[-\infty\ to\ \infty] = \frac{\frac{1}{\sqrt{2\pi}}\exp\{-\frac{\left[\frac{\exp(1)-x}{x[\exp(1)-1]}\right]^2}{2}\}}{\left\|\frac{\exp(1)[\exp(1)-1]}{\left\{1+\frac{\exp(1)-x}{x[\exp(1)-1]}[\exp(1)-1]\right\}^2}\right\|} \tag{20}$$

The comparison between the result of proposed method and analytical solution is shown in Figure 2. It shows the results by the proposed method accord almost exactly with the analytical solution.

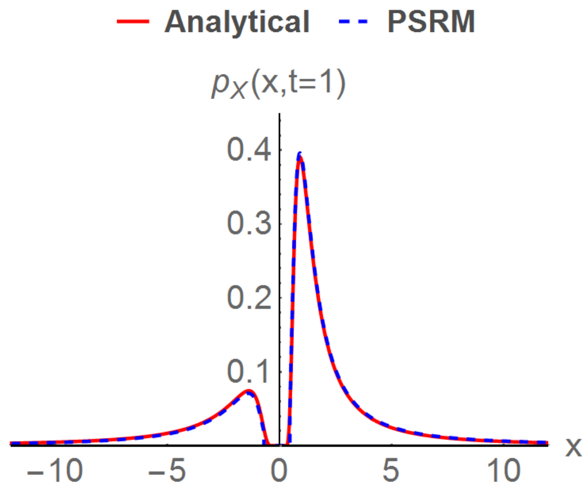

Figure 2: Comparison between the theoretical solution and PSRM result for the Riccati system.

4.2 A SDOF oscillator

The equation of this oscillator reads

$$\ddot{X} + \omega^2 X = 0 \tag{21}$$

with the initial condition $X(0) = 0.1, \dot{X}(0) = 0$.

The random variable ω is uniformly distributed over the interval $[5\pi/4, 7\pi/4]$. The analytical solution of the problem is [6]

$$X(\omega, t) = 0.1\cos(\omega t) \tag{22}$$

The theoretical results of the system and the solution of the proposed method are shown in Figure 3. Clearly, the two curves accords almost exactly.

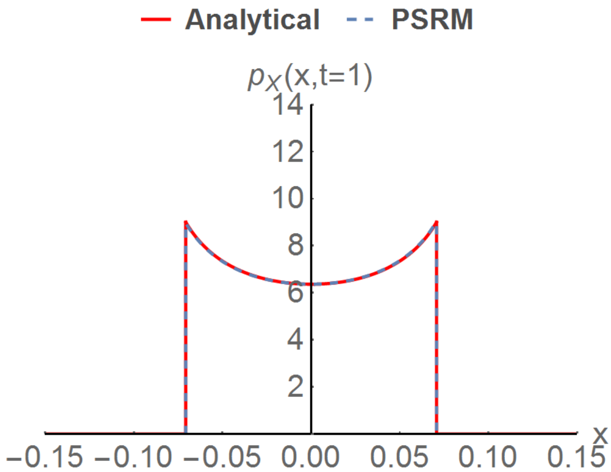

Figure 3: Comparison between the theoretical solution and PSRM for the SDOF system.

4.3 A Duffing oscillator

The equation of this oscillator reads

$$\ddot{X} - \theta^2 \cdot (X - X^3) = 0 \tag{23}$$

with the initial condition $X(0) = 0, \dot{X}(0) = 1$.

The random variable θ is uniformly distributed over the interval $[0,5]$. 10^5 times of Monte Carlo simulation are carried on to verify the proposed method. The comparison between the Monte Carlo simulation result and the PSRM solution is shown in Figure 4. Again, it is seen that the proposed method is of high accuracy.

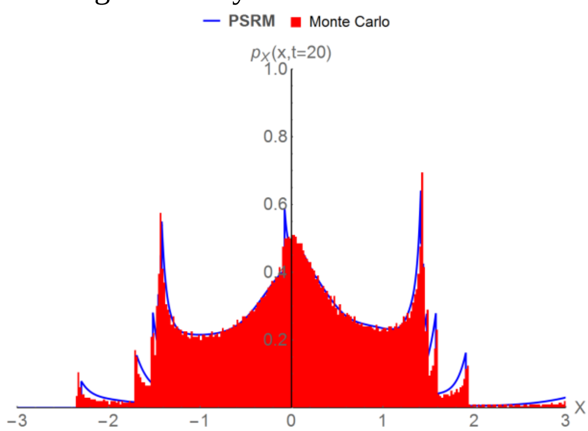

Figure 4: Comparison between the Monte Carlo simulation and PSRM for the Duffing system.

5 CONCLUSIONS

- A new method named phase space reconstruction method (PSRM) is proposed to solve the generalized density evolution equation (GDEE). With this approach, the instantaneous PDF of the dynamic response of non-linear systems is obtainable with high accuracy and efficiency.
- Three typical non-linear SDOF systems with random parameters, including a Riccati oscillator, a linear SDOF system and a Duffing oscillator, are studied as examples. The results are compared with the analytical solution or the Monte Carlo simulation results, verifying the accuracy of the proposed method.
- More complex multi-dimensional stochastic systems with multiple basic random variables are to be studied in the future investigations.

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