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A FIRST ORDER ASSEMBLE EVOLUTION METHOD FOR SOLVING GDEE

Jie Li[1], Wei-Feng Tao[2]

[1] College of Civil Engineering, Tongji University
Shanghai, China
e-mail: lijie@tongji.edu.cn

[2] College of Civil Engineering, Tongji University
Shanghai, China
e-mail: 1040020008@tongji.edu.cn

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Abstract. *Probability density evolution method (PDEM) provides a feasible approach for stochastic response analysis of general structures. The key issue of PDEM is to solve a generalized density evolution equation (GDEE). Previously, GDEE was solved in the framework of point evolution method which is in essence a zero order assemble evolution method. In this paper, a first order assemble evolution method is proposed aiming at increasing the accuracy and robustness of PDEM. The main idea of the proposed method is to extend the velocity term of the derived density evolution equation from integration of GDEE in a subdomain of the random space with a first order term with respect to the response variable. Compared to the original one, the proposed method directly introduces inherent diffusivity into PDEM which is essential in order to better reflect the evolution result of the corresponding subdomain. The same as in the point evolution case, TVD finite difference scheme is adopted to solve the new derived density evolution equation. A single-degree-of-freedom (SDOF) oscillator with random frequency is studied in the paper, and the result indicates that the proposed method performs better than the original one.*

1 INTRODUCTION

Deterministic finite element method has been studied in depth, and is widely applied in practical engineering problems to date. On the other hand, there are a lot of inevitable randomness in many aspects of structure analysis, such as material properties, geometric sizes, boundary conditions and applied loads. Variability of these parameters may lead to considerable fluctuation in structural response, and may even result in structural failure. In order to trace the propagation of randomness and measure its influence on structural response, it is necessary to resort to stochastic finite element method (SFEM).

Study on SFEM was initiated in 1970's. Over the past four decades, great developments have been achieved in this field. Available approaches by far can be broadly divided into Monte Carlo method (MCM), Perturbation stochastic finite element method (PSFEM), and orthogonal polynomial expansion method (OPEM). Pioneering work to introduce MCM into stochastic structural analysis was conducted by Shinozuka and Jan [1]. Despite its versatility, MCM is seldom applied to complex structures due to its prohibitive computational consumption. In order to increase the efficiency of MCM, Shinozuka and Deodatis [2] and Yamazaki, Member and Shinozuka [3] introduced the Neumann expansion technique so that the time-consuming matrix inversion operation in each random sampling is avoided. Hisada and Nakagiri [4] proposed PSFEM by using stochastic perturbation technique to deal with fluctuation of random parameters. Frangopol, Lee and Willam [5] carried out nonlinear static analysis of concrete frame structures and concrete plates by combining PSFEM with concrete constitutive law. First order PSFEM is rather effective on condition that the random parameters are of very small variability. Second order PSFEM has looser limit on the variability of the random parameters, but is of very low efficiency and thus not suitable for practical engineering applications. Moreover, the secular terms problem (Liu, Besterfield and Belytschko [6]) will result in large error when applied to dynamic problems. Sun [7] proposed a new class of numerical algorithm to expand the structural response by Hermit orthogonal polynomial when trying to solve random differential equations with random coefficients. Enlightened by Sun's work, Jensen and Iwan [8, 9] applied OPEM into structural seismic response analysis. Ghanem and Spanos [10] suggested to use chaos polynomial as orthogonal basis. By applying the sequential orthogonal decomposition principle in the random space, Li [11, 12] independently derived an extended system method. OPEM is not troubled by the limit on the variability of random parameters and the secular terms problem. However, when the number of random parameters is large, order of the extended system will be extremely higher than that of the original one, thus making the computational consumption almost unbearable.

Recently, Li and Chen [13, 14] developed a new class of PDEM, where they derived a GDEE based on the principle of preservation of probability. The point evolution method was usually adopted to obtain the evolution of probability density function (PDF) of the specific dynamic response. In this paper, a first order assemble evolution method is proposed in order to increase the accuracy and robustness of PDEM and an SDOF oscillator with random frequency is studied.

2 GENERALIZED DENSITY EVOLUTION EQUATION

Without loss of generality, consider a general stochastic dynamic system as follows

$$\dot{\boldsymbol{X}} = G(\boldsymbol{X}, \boldsymbol{\Theta}, t), \quad \boldsymbol{X}(t_0) = \boldsymbol{x}_0 \tag{1}$$

where $\boldsymbol{X}=(X_1,\cdots,X_n)^{\mathrm{T}}$ is a n-dimensional vector of system responses; $\boldsymbol{\Theta}=(\theta_1,\cdots,\theta_s)^{\mathrm{T}}$ is a s-dimensional vector of random parameters, the joint probability density function (JPDF) of which is $p_{\boldsymbol{\Theta}}(\boldsymbol{\theta})$. For any well-posed dynamic system, there exists a unique solution to Eq. (1)

$$\boldsymbol{X}=H(\boldsymbol{\Theta},t) \tag{2}$$

the component form of which is

$$X_l=H_l(\boldsymbol{\Theta},t) \quad l=1,\cdots,n \tag{3}$$

Based on this, a GDEE can be derived according to the random event description of the principle of preservation of probability

$$\frac{\partial p_{\boldsymbol{X}\boldsymbol{\Theta}}(\boldsymbol{x},\boldsymbol{\theta},t)}{\partial t}+\sum_{l=1}^{n}\dot{H}_l(\boldsymbol{\theta},t)\cdot\frac{\partial p_{\boldsymbol{X}\boldsymbol{\Theta}}(\boldsymbol{x},\boldsymbol{\theta},t)}{\partial x_l}=0 \tag{4}$$

where $p_{\boldsymbol{X}\boldsymbol{\Theta}}(\boldsymbol{x},\boldsymbol{\theta},t)$ is the JPDF of $(\boldsymbol{X},\boldsymbol{\Theta})$. If $n=1$, then Eq. (4) simplifies to a one-dimensional GDEE

$$\frac{\partial p_{X\boldsymbol{\Theta}}(x,\boldsymbol{\theta},t)}{\partial t}+\dot{H}(\boldsymbol{\theta},t)\cdot\frac{\partial p_{X\boldsymbol{\Theta}}(x,\boldsymbol{\theta},t)}{\partial x}=0 \tag{5}$$

with the initial condition

$$p_{X\boldsymbol{\Theta}}(x,\boldsymbol{\theta},t_0)=\delta\left(x-H(\boldsymbol{\theta},t_0)\right)\cdot p_{\boldsymbol{\Theta}}(\boldsymbol{\theta}) \tag{6}$$

where the subscript l is omitted for brevity. Meanwhile, for convenience of discussion, the paper is only limited to one-dimensional GDEE.

3 ASSEMBLE EVOLUTION METHOD

3.1 Theoretical derivation

In theory, there is a formal solution to Eq. (5) with the initial condition Eq. (6)

$$p_{X\boldsymbol{\Theta}}(x,\boldsymbol{\theta},t)=\delta\left(x-H(\boldsymbol{\theta},t)\right)\cdot p_{\boldsymbol{\Theta}}(\boldsymbol{\theta}) \tag{7}$$

where $\delta(\cdot)$ is the Dirac δ function. Integrate both sides of Eq. (7) with respect to $\boldsymbol{\theta}$ in the random space $\boldsymbol{\Omega}$

$$p_X(x,t)=\int_{\boldsymbol{\Omega}}p_{X\boldsymbol{\Theta}}(x,\boldsymbol{\theta},t)d\boldsymbol{\theta}=\int_{\boldsymbol{\Omega}}\delta\left(x-H(\boldsymbol{\theta},t)\right)\cdot p_{\boldsymbol{\Theta}}(\boldsymbol{\theta})d\boldsymbol{\theta} \tag{8}$$

However, since $H(\boldsymbol{\theta},t)$ generally does not have an explicit expression, it is impractical to obtain $p_X(x,t)$ by way of Eq. (8). Hence, it is necessary to resort to an approximate method. Suppose that the random space $\boldsymbol{\Omega}$ is divided uniformly into N_{sel} disjoint and complementary subdomains $\boldsymbol{\Omega}_q\,(q=1,\cdots,N_{sel})$, and each subdomain corresponds to a representative point $\boldsymbol{\theta}_q$ located in its center. Integrate both sides of Eq. (5) with respect to $\boldsymbol{\theta}$ in each subdomain

$$\int_{\boldsymbol{\Omega}_q}\frac{\partial p_{X\boldsymbol{\Theta}}(x,\boldsymbol{\theta},t)}{\partial t}d\boldsymbol{\theta}+\int_{\boldsymbol{\Omega}_q}\dot{H}(\boldsymbol{\theta},t)\cdot\frac{\partial p_{X\boldsymbol{\Theta}}(x,\boldsymbol{\theta},t)}{\partial x}d\boldsymbol{\theta}=0 \quad q=1,\cdots,N_{sel} \tag{9}$$

For the first term on the left side of the equation above, exchanging the order of derivation and integration yields

$$\int_{\Omega_q} \frac{\partial p_{X\Theta}(x,\boldsymbol{\theta},t)}{\partial t} d\boldsymbol{\theta} = \frac{\partial}{\partial t} \int_{\Omega_q} p_{X\Theta}(x,\boldsymbol{\theta},t) d\boldsymbol{\theta} = \frac{\partial p_X^{(q)}(x,t)}{\partial t} \tag{10}$$

For the second term, making use of the mean value theorem of integrals yields

$$\begin{aligned} \int_{\Omega_q} \dot{H}(\boldsymbol{\theta},t) \cdot \frac{\partial p_{X\Theta}(x,\boldsymbol{\theta},t)}{\partial x} d\boldsymbol{\theta} &= \dot{H}\left(\vartheta_q(x,t),t\right) \int_{\Omega_q} \frac{\partial p_{X\Theta}(x,\boldsymbol{\theta},t)}{\partial x} d\boldsymbol{\theta} \\ &= \dot{H}\left(\vartheta_q(x,t),t\right) \frac{\partial p_X^{(q)}(x,t)}{\partial t} \end{aligned} \tag{11}$$

It should be pointed out that in most cases $\vartheta_q(x,t)$ does not have an explicit expression. Substitute Eq. (10) and Eq. (11) into Eq. (9)

$$\frac{\partial p_X^{(q)}(x,t)}{\partial t} + \dot{H}\left(\vartheta_q(x,t),t\right) \frac{\partial p_X^{(q)}(x,t)}{\partial t} = 0 \qquad q = 1,\cdots,N_{sel} \tag{12}$$

where $\dot{H}\left(\vartheta_q(x,t),t\right)$ is an unknown function of x and t. Since each subdomain is very small compared to the whole random space, it is reasonable to assume that $p_X^{(q)}(x,t)$ is a unimodal and symmetrically distributed function, which has been confirmed by plenty of computation. Therefore, in order to get an approximate expression of $\dot{H}\left(\vartheta_q(x,t),t\right)$, suppose that the solution to Eq. (12) is a Gaussian function with mean value $m_q(t)$ and standard deviation $\sigma_q(t)$

$$p_X^{(q)}(x,t) = \frac{1}{\sqrt{2\pi}\sigma_q(t)} e^{-\frac{\left(x-m_q(t)\right)^2}{2\sigma_q^2(t)}} \tag{13}$$

Substitute Eq. (13) into Eq. (12), then the following expression can be obtained after a simple derivation

$$\dot{H}\left(\vartheta_q(x,t),t\right) = \dot{m}_q(t) + \dot{\sigma}_q(t)\left(\frac{x-m_q(t)}{\sigma_q(t)} - \frac{\sigma_q(t)}{x-m_q(t)}\right) \tag{14}$$

As a reasonable approximation, truncating the right side of the equation above to the second term and substituting it into Eq. (12) yields

$$\frac{\partial p_X^{(q)}(x,t)}{\partial t} + \left(\dot{m}_q(t) + \frac{\dot{\sigma}_q(t)}{\sigma_q(t)} \cdot \left(x - m_q(t)\right)\right) \frac{\partial p_X^{(q)}(x,t)}{\partial t} = 0 \qquad q = 1,\cdots,N_{sel} \tag{15}$$

with the corresponding initial condition

$$p_X^{(q)}(x,t_0) = P_q \cdot \delta\left(x - m_q(t_0)\right) \qquad q = 1,\cdots,N_{sel} \tag{16}$$

Eq. (15) can be termed a derived density evolution equation. Its coefficient on the left side is composed of two terms: the first term, being a function of t, dominates the law of drift of the stochastic system response and is called point velocity while the second term, being a function of both x and t, dominates the law of fluctuation and is called assemble velocity.

3.2 Numerical algorithm

Eq. (15) is a first order linear hyperbolic equation, and can be solved by finite difference method. The solving procedure is as follows

1. Select a certain amount of representative points $\boldsymbol{\theta}_q \left(q = 1, \cdots, N_{sel}\right)$ that are uniformly distributed in the random space $\boldsymbol{\Omega}$ and calculate the corresponding assigned probabilities P_q (Li and Chen [15]);
2. Compute $m_q(t)$ and $\sigma_q(t)$ that correspond to each subdomain $\boldsymbol{\Omega}_q \left(q = 1, \cdots, N_{sel}\right)$;
3. Generate mesh and approximate Eq. (16) as

$$p_X^{(q)}\left(x_j, t_0\right) = \begin{cases} \dfrac{P_q}{\Delta x} & x_j \in \left[m_q\left(t_0\right) - 0.5\Delta x, m_q\left(t_0\right) + 0.5\Delta x\right) \\ 0 & \text{otherwise} \end{cases} \qquad q = 1, \cdots, N_{sel} \tag{17}$$

   where $x_j = j\Delta x \left(j = 0, \pm 1, \cdots\right)$, Δx is the mesh size on the *x*-axis;
4. Solve Eq. (15) by finite difference method to obtain $p_X^{(q)}\left(x_j, t_k\right)$, where $t_k = k\Delta t \left(k = 0, 1, \cdots\right)$, Δt is the time step;
5. Add all $p_X^{(q)}\left(x_j, t_k\right)$ together to get the final result

$$p_X\left(x_j, t_k\right) = \sum_{q=1}^{N_{sel}} p_X^{(q)}\left(x_j, t_k\right) \tag{18}$$

Among all the steps above, step 2 plays an important role in deciding the accuracy of the final result. A feasible method to compute $m_q(t)$ and $\sigma_q(t)$ is to calculate the response time histories corresponding to the representative points and densify the whole random space by way of interpolation or regression. Once the random space is densified, it is quite straight to compute $m_q(t)$ and $\sigma_q(t)$ by definition. In this paper, since the numerical example given is only a one-dimensional problem, it is sufficient to densify the random space by linear interpolation.

4 NUMERICAL EXAMPLE

In order to verify the feasibility of the proposed method as well as to compare it with the original point evolution method, an SDOF oscillator with random frequency is studied here.

Equation of motion of free vibration of an undamped SDOF oscillator is

$$\ddot{X}(t) + \omega^2 X = 0 \tag{19}$$

with deterministic initial condition

$$X(t)\big|_{t=0} = 0.1\text{m}, \dot{X}(t)\big|_{t=0} = 0 \tag{20}$$

In analysis, the natural frequency ω is assumed to be a normally distributed random variable with mean value 3π and coefficient of variation 0.1. The analytical solution for the time dependent PDF of displacement can refer to Li and Chen [13]. Both the point evolution method and first order assemble evolution method are applied and the results are depicted in Figure 1 to Figure 3. Depicted in Figure 1 and Figure 2 are typical time histories of mean value and standard deviation, respectively. It is interesting to see that the standard deviation exhibits obvious fluctuation with time, which cannot be captured by the point evolution method. Depicted in Figure 3 are PDFs at $t = 0.65\text{sec}$ by analytical method, the point evolution method and the first order assemble evolution method. It is obvious that the result by the proposed method

fits very well with the analytical solution while the result by the point evolution method oscillates heavily.

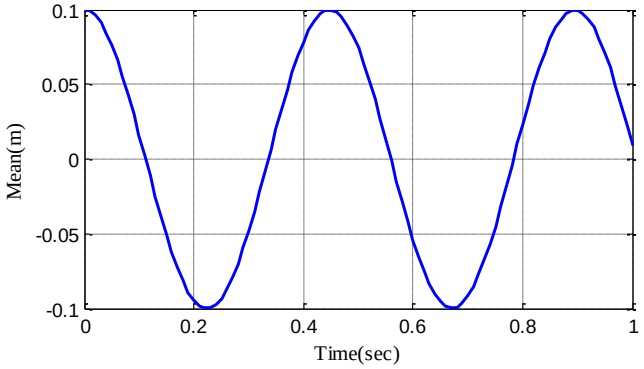

Figure 1: Typical time history of mean value

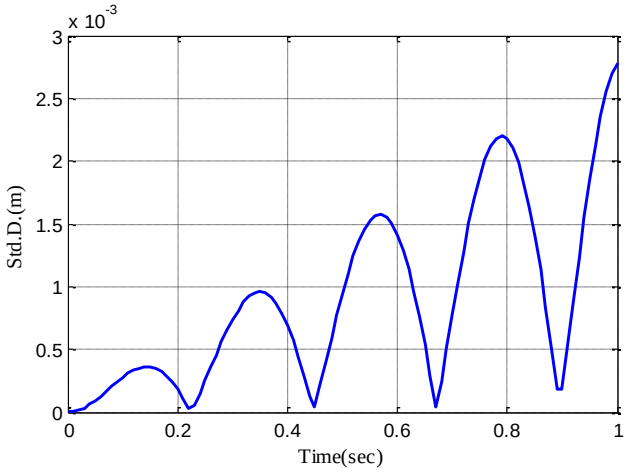

Figure 2: Typical time history of standard deviation

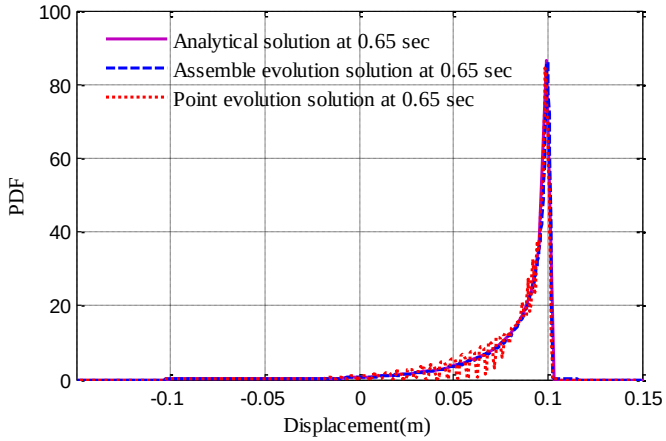

Figure 3: Comparison of three PDFs at $t = 0.65\,\mathrm{sec}$

5 CONCLUSIONS

A first order assemble evolution method for solving GDEE is proposed in this paper. Compared to the point evolution method, the new method introduces an assemble velocity

term which dominates the law of fluctuation of the stochastic system response into the derived density evolution equation. An SDOF oscillator with random frequency is studied, and the result indicates that the proposed method performs better than the original one.

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