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THE COMMON BASIS FOR PROBABILISTIC AND NON-PROBABILISTIC DESIGN OF STRUCTURES UNDER DIFFERENT DEGREES OF DATA AVAILABILITY

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Abstract. *Non-probabilistic design of structures is a matter of assigning values to the uncertain parameters involved in the design problem from an often closed, bounded and convex set. The mathematical problem is then to ensure that the prescribed safety constraints are satisfied for all the points of this set. Probabilistic design of structures on the other hand works with assigning probability distributions to the uncertain parameters involved and by requiring that the safety constraints are satisfied with high probabilities. Through an ellipsoidal representation of uncertainty, the link between these methodologies is illustrated and discussed in this article. Specific focus is placed on scenarios where due to lack of underlying data, only the first and second moment information can be established, but where the use of parametric probability distributions might be controversial. The recent generalizations of the classical Chebychev and Gauss inequalities are utilized in this respect. A numerical example is used for illustration.*

1 PROBABILISTIC VERSUS ROBUST OPTIMIZATION OF STRUCTURES

1.1 Introduction

Designing of any mechanical system or structure can be interpreted and mathematically formulated as an optimization problem. Design variables and parameters usually include cross-sectional properties of the structural members, material properties, structural geometry, boundary conditions and the applied loads. Some of these quantities are usually taken as deterministic design variables whose values have to be decided by the designer. Others are treated as parameters that are subject to uncertainty, to be considered in the design process. The rest are assumed to be parameters deterministically known to the designer. These are the parameters whose uncertainty, if at all, does not considerably affect the design based on previous experience or pre-engineering. The problem can be formulated in various ways but the ultimate goal would be to design a structure that satisfies a number of safety-related constraints. Such constraints can relate to functional, ultimate, accidental and fatigue requirements in various applications.

Considering the uncertainties in the design parameters, specifically in the material and load properties is a central issue in any structural design and has been the subject of comprehensive research. The most well-known approach is to treat the uncertainties probabilistically by assigning (joint) probability distributions to the design parameters and targeting to satisfy a predefined probability of structural failure. Numerous papers, manuscripts and books have been written on the subject of structural reliability and probabilistic mechanics, e.g. see [1] and its bibliography for a review of the field. In a broader context, decision making under probabilistically-represented uncertainty has been widely studied across disciplines such as mathematical optimization, finance, and operations research. The term stochastic optimization is generally used by the latter communities to describe an optimization problem where some of its parameters are probabilistically described. Stochastic optimization dates as far back as Dantzig [2] and has seen enormous developments in the past few decades. Ingredients of stochastic optimization literature have lots in common with structural reliability literature. The interested reader is referred to the several textbooks and articles that have been written on stochastic optimization, such as [3] and [4] and the references therein for a comprehensive review of the field.

The second alternative for considering the uncertainties in the design parameters is to associate them with a deterministic set. In other words, the design parameters being integrated parts of the design optimization problem are assumed to come from a set (with dimension of uncertainty) which is often taken to be closed, bounded and convex. In general, the geometry of the uncertainty sets can be described using different closed surfaces such as ellipsoidal, polyhedral and hypercubical sets which are all convex. Convexity is not a strict requirement for the uncertainty sets, but it is often an appealing mathematical property that brings about unique computational benefits. Ben-Haim and Elishakoff [5] in a manuscript entitled convex models of uncertainty in applied mechanics were among the first who systematically introduced this approach to the structural and mechanical engineering communities. This approach has ever since been referred to as a non-probabilistic method, convex modeling, or a set-theoretic approach by the researchers of the respective fields. Other disciplines have also been in parallel researching problems of a similar mathematical nature, applied to different applications. Robust optimization is a widely used terminology by mathematical optimization, finance and operations research communities to describe this type of problem from the mathematical standpoint. In principle, a (deterministically) robust optimization is an optimization problem that has been immunized against change of its parameters, described by particular closed sets.

The book by Ben-Tal et al. [6] is a modern and advanced book that provides an extensive presentation of the state of the art of this discipline. The term semi-infinite programming has also been widely used by other groups of researchers to describe similar types of problems, e.g. see the review by Shapiro [7]. The preferred terminology in this article is robust optimization.

So-called non-probabilistic methods in civil and mechanical engineering are often advocated to suit the scenarios of data scarcity, as a replacement, a competitor and sometimes a companion for probabilistic analysis. In scenarios with lack of data, it is claimed that parametric probability distributions either cannot be established or remain a matter of hesitation. As a result, the probabilistic analysis based on such distributions are viewed with concern and even disbelief. As an example, long-lasting discussions exist in the literature criticizing the tail sensitivity of reliability analyses. Whether or not such arguments are always relevant (e.g. they can be a result of old-fashioned malpractices in statistical analysis), the issue of scare data is a major concern. One may however argue against non-probabilists that the issue of data scarcity is more essential than to be resolvable by a change of methodology. Determining the shape and size of the uncertainty set in non-probabilistic or robust optimization can be envisaged as controversial as determining the type of probability distributions in probabilistic analysis. For instance, in working with experimental measurements, wrapping an ellipsoidal or hypercubical surface around some measured data based on a minimum volume philosophy (e.g. see [8] and [9]) can be readily controversial. Such a practice can suffer from two disadvantages. The first is that it entirely rules out the possibility that another set of measurements falls outside the fitted bounded set. The second can be that the exterior points (outliers in statistical terminology) are given too much weight, as the shape and size of the set e.g. an ellipsoid, is unrightfully highly influenced by these data points. Let us put aside the situation where quantities such as mechanical property (e.g. modulus of elasticity) measurements are concerned, where the design value is at least in a foreseeable range of the measured data. When environmental phenomena such as wind, wave and sea ice are concerned, our measurements of their characterizing parameters can only span duration of a few years. It is however often required to design the structures in these environments for much longer return periods as compared to the measured data. Therefore, it is virtually impossible to adopt a methodology of wrapping around an ellipsoidal or a hypercubical surface around the measured data points, which are supposedly much smaller in magnitude than the ones resulting in anticipated design loads. For instance, when inferring the 100-year performance of an offshore platform given few years of environmental measurements, the power of probability theory (e.g. covariance relations as a basis for statistical inference) should not be neglected, which can be used as the most consistent basis of establishing the uncertainty set. Therefore, the approach to tackle such problems cannot be non-probabilistic in terms of establishing the uncertainty set, but it can be so when it comes to the form of the optimization problem and the methodology for treating it. In addition, many modern design standards e.g. in offshore engineering have adopted probabilistic design philosophies which require the satisfaction of a minimum target reliability. This also makes the use of a purely non-probabilistic method impossible in practice. In this case, even if the size of the uncertainty set is decided, and the possibly challenging mathematical labor of performing the robust optimization is carried out, one still faces the question whether the code requirements are satisfied.

Robust (non-probabilistic, or possibilistic) optimization of structures is an interesting field that has all the rights of being investigated-as legitimately done so (with exceptions) by many researchers in the past. However, in this article, a shoulder-to-shoulder presentation of probabilistic and non-probabilistic analyses is provided, with specific focus on how they are interrelated. This way, almost any robust design optimization will have an immediate probabilistic interpretation, depending on the strength of assumptions that can be made about the probability distribution of the underlying data, from inclusive to very mild. For decades, the probabilistic design optimization (and the lower level reliability analysis) has been borrowing from the robust optimization philosophy in adopting analytical methods such as FORM and SORM. The opposite is also possible. The non-probabilistic design optimization, developed separately by the structural and mechanical engineering communities and surviving almost independently, can in return benefit from and be related to probabilistic analysis. This is nothing new in many disciplines but has probably not received enough attention in structural and mechanical engineering literature. Few exceptions exist but they mainly deal with the specific and very limited case of uniform distributions (either on ellipsoidal or hypercubical support) where the probabilistic and non-probabilistic analyses are shown to be the same in the limit, i.e. when the reliability approaches to one e.g. see [10]. The present paper contains elements inspired by Langley's work [11]. However, as indicated, the focus of the present paper is on the link between stochastic optimization (that is, structural optimization subject to a probabilistic constraint – for example, minimizing weight whilst maintaining a specified maximum failure probability) and robust optimization (that is, structural optimization subject to a robust constraint – for example, minimizing weight whilst maintaining structural integrity over a specified set of possible realizations of the uncertain parameters). By adopting an ellipsoidal uncertainty description for the robust optimization problem, but also by allowing the data to be characterized statistically by averages and a covariance matrix, the link between the two approaches is found to involve the covariance matrix, the ellipsoidal configuration matrix and an ellipsoidal size measure. Specifically, the latter is a function of the specified maximum failure probability, and this function is obtained via generalizations of probability inequalities (such as the Chebychev and Gauss inequalities). The key result of this is that a possibilistic design (in the robust optimization sense) may still be performed based on a specified maximum failure probability: despite the lack of a fully specified joint probability distribution function, the probability inequalities allow the specified maximum failure probability to be linked to the size of the uncertainty ellipsoid required for the possibilistic design.

1.2 Mathematical Formulation

The reliability-based or probabilistic optimization of a structure can be formulated as in equation (1). The formulation is a minimization with respect to design variables $\mathbf{x}$ where the objective function $f(\mathbf{x})$ is usually an indicator of structural weight or cost. The constraint, being a function of both the deterministic design variables $\mathbf{x}$ and the uncertain design parameters $\mathbf{u}$, requires that the structural reliability should exceed the target reliability level of $\eta = 1 - \varepsilon$ or that the structural failure be smaller than a target failure probability of ε. In other words, it is required that the inequality constraint $h(\mathbf{x}, \mathbf{u}) \leq 0$, which is an indicator of structural safety, be held with a given and often high probability. The constraint in equation (1) is called a chance constraint in stochastic optimization literature. A single chance constraint is considered here which indicates that the structural behavior can be described by one equation, which is often called a component reliability problem in structural reliability literature.

$$\begin{aligned} &\underset{\mathbf{x}}{\text{minimize}} \;\; f(\mathbf{x}) \\ &\text{subject to} \quad \text{Prob}(\underbrace{h(\mathbf{x}, \mathbf{u}) \leq 0}_{Safe\ State}) \geq \underbrace{1 - \varepsilon}_{\eta} \end{aligned} \tag{1}$$

The non-probabilistic or robust optimization of a structure can be formulated as in equation (2). Here, a safe state is required for all values of the uncertain design parameters $\mathbf{u}$ belonging to the set $\Theta(\mathbf{u})$.

$$\begin{aligned} &\underset{\mathbf{x}}{\text{minimize}} \;\; f(\mathbf{x}) \\ &\text{subject to} \quad \underbrace{h(\mathbf{x},\mathbf{u}) \leq 0}_{Safe\ State} \quad \forall \mathbf{u} \in \Theta(\mathbf{u}) \end{aligned} \tag{2}$$

A common technique to attempt to solve problem (2) is to treat it in two levels. This implies:

$$\begin{aligned} &\underset{\mathbf{x}}{\text{minimize}} \;\; f(\mathbf{x}) \\ &\text{subject to} \quad \tilde{h}(\mathbf{x}) \leq 0 \end{aligned} \tag{3}$$

Where $\tilde{h}(\mathbf{x})$ is given by,

$$\tilde{h}(\mathbf{x}) = \sup\left\{ h(\mathbf{x},\mathbf{u}) \mid \mathbf{u} \in \Theta(\mathbf{u}) \right\} \tag{4}$$

Algorithmically, we can start by an initial $\mathbf{x}$. Then, we postulate that in order for the robust constraint in (2) to hold for all $\mathbf{u} \in \Theta(\mathbf{u})$, it is required that it holds for the worst (or supremum) constraint given that particular $\mathbf{x}$. The supremum constraint is then calculated by a lower level optimization problem that maximizes $h(\mathbf{x},\mathbf{u})$, constrained by $\mathbf{u} \in \Theta(\mathbf{u})$ (see (4)). The membership in $\Theta(\mathbf{u})$ is in practice often described by inequalities. A higher level iteration on $\mathbf{x}$ is performed until a solution is found. Various numerical algorithms can be used in order to (attempt to) solve the problem. Tractable versions of the problem exist, often requiring strong assumptions on $h(\mathbf{x},\mathbf{u})$ (such as its bi-affinity with respect to both $\mathbf{x}$ and $\mathbf{u}$) and $f(\mathbf{x})$ (its affinity).

In this paper, the safe state is presented using a negative inequality $h(\mathbf{x},\mathbf{u}) \leq 0$. However, the well-known safety margin in structural reliability literature, let's call it $g(\mathbf{x},\mathbf{u})$, often takes an opposite sign, i.e. $g(\mathbf{x},\mathbf{u}) \geq 0$ indicates safety and $g(\mathbf{x},\mathbf{u}) \leq 0$ indicates failure. It is apparent that the conversion $h(\mathbf{x},\mathbf{u}) = -g(\mathbf{x},\mathbf{u})$ can be trivially made. The border between safety and failure $g(\mathbf{x},\mathbf{u}) = 0$ is called the failure surface which is the same as $h(\mathbf{x},\mathbf{u}) = 0$. The convention of working with $h(\mathbf{x},\mathbf{u}) \leq 0$ has been made in this paper to remain consistent with the optimization literature where constraints are usually formally presented as negative inequalities. This assumption makes it less confusing to read the underlying optimization literature for the interested reader. For instance, it leads to familiar *min-max* (or *min-sup*), rather than the unfamiliar *min-min* (or *min-inf*) bi-level formulations of optimization problems with uncertainty (as for instance discussed in (3) and (4)).

2 PROBABILISTIC OPTIMIZATIONS AND THEIR ROBUST COUNTERPARTS

We start by considering the case where the function $h(\mathbf{x},\mathbf{u})$ appearing in stochastic and robust formulations (1) and (2) is affine (or linear) with respect to $\mathbf{u} \in \mathbb{R}^{nu}$. We denote $h(\mathbf{x},\mathbf{u}) = \mathbf{a}^T\mathbf{u} + b$ with $\mathbf{a} \in \mathbb{R}^{nu}$ and $b \in \mathbb{R}$. Coefficients $\mathbf{a}$ and b are functions of the design variables $\mathbf{x} \in \mathbb{R}^{nx}$, or $\mathbf{a} = a(\mathbf{x})$ and $b = b(\mathbf{x})$.

We first investigate the robust optimization problem (2). Assuming an ellipsoidal description of uncertainty for $\mathbf{u}$, the optimization problem is written as:

$$\begin{aligned} &\underset{\mathbf{x}}{\text{minimize}} \;\; f(\mathbf{x}) \\ &\text{subject to} \quad a(\mathbf{x})^T \mathbf{u} + b(\mathbf{x}) \le 0 \quad \forall \mathbf{u} \in \Xi(\mathbf{u}) = \left\{ \mathbf{u}_0 + \mathbf{P}t \mid \left\| t \right\|_2 \le 1 \right\} \end{aligned} \tag{5}$$

Where $\Xi(\mathbf{u})$ is the description of an ellipsoidal set, with $\mathbf{u}_0 \in \mathbb{R}^{nu}$ being the center and $\mathbf{P} \in \mathbb{R}^{nu \times nu}$ being a symmetric positive definite matrix determining the shape and size of the ellipsoid i.e. $\mathbf{P} = \mathbf{P}^T \succ 0$. The curved inequality symbol indicates the positive definiteness. In fact, $\mathbf{u}_0 + \mathbf{P}t$ is an affine transformation of the unit ball $\left\| t \right\|_2 \le 1$, which creates an ellipsoid. The Euclidean norm (or the 2-norm) is defined as: $\left\| t \right\|_2 = \sqrt{t^T t}$. The robust constraint in (5) is equivalent to:

$$\sup \left\{ a(\mathbf{x})^T \mathbf{u} + b(\mathbf{x}) \mid \mathbf{u} \in \Xi = \left\{ \mathbf{u}_0 + \mathbf{P}t \mid \left\| t \right\|_2 \le 1 \right\} \right\} \le 0 \tag{6}$$

In other words, in order for the constraint $a(\mathbf{x})^T \mathbf{u} + b(\mathbf{x}) \le 0$ to hold for all values of $\mathbf{u}$ coming from the ellipsoid Ξ, it is as if we require that the supremum of $a(\mathbf{x})^T \mathbf{u} - b(\mathbf{x})$ constrained by $\mathbf{u} \in \Xi$ is negative (see previous formulations (3) and (4)). Equation (6) then converts to:

$$a(\mathbf{x})^T \mathbf{u}_0 + \sup \left\{ t^T \mathbf{P}^T \mathrm{a}(\mathbf{x}) \mid \left\| t \right\|_2 \le 1 \right\} + b(\mathbf{x}) \le 0 \tag{7}$$

Which becomes,

$$a(\mathbf{x})^T \mathbf{u}_0 + \left\| \mathbf{P}^T a(\mathbf{x}) \right\|_2 + b(\mathbf{x}) \le 0 \tag{8}$$

Equation (8) is a deterministic analogue of the linear robust constraint in (5). This is a classical result in robust optimization. Reference is for instance made to [12] and the references therein.

We now study the probabilistic optimization problem (1), given the linear safety indictor function $a(\mathbf{x})^T \mathbf{u} + b(\mathbf{x}) \le 0$. The optimization problem reads:

$$\begin{aligned} &\underset{\mathbf{x}}{\text{minimize}} \;\; f(\mathbf{x}) \\ &\text{subject to} \quad \mathrm{Prob}(a(\mathbf{x})^T \mathbf{u} + b(\mathbf{x}) \le 0) \ge 1 - \varepsilon \end{aligned} \tag{9}$$

We assume that $\mathbf{u}$ belongs to an ambiguous family of distributions $\mathbf{u} \sim \mathbb{Q}(\bar{\mathbf{u}}, \Sigma)$ with known means and covariances $\bar{\mathbf{u}}$ and $\mathbf{\Sigma}$. In other words, $\mathbb{Q}$ describes any (joint) probability distribution that satisfies the first and second moment information of the underlying data. The stochastic constraint in (9) is equivalent to:

$$\inf_{\mathbf{u} \sim \mathbb{Q}(\bar{\mathbf{u}}, \mathbf{\Sigma})} \left\{ \mathrm{Prob}(a(\mathbf{x})^T \mathbf{u} + b(\mathbf{x}) \le 0) \right\} \ge 1 - \varepsilon \tag{10}$$

Or,

$$\sup_{\mathbf{u} \sim \mathbb{Q}(\bar{\mathbf{u}}, \mathbf{\Sigma})} \left\{ \mathrm{Prob}(a(\mathbf{x})^T \mathbf{u} + b(\mathbf{x}) \ge 0) \right\} \le \varepsilon \tag{11}$$

By treating the expected value and variance of the random variable $a(\mathbf{x})^T \mathbf{u}$ as $E(a(\mathbf{x})^T \mathbf{u}) = a(\mathbf{x})^T E(\mathbf{u}) = a(\mathbf{x})^T \bar{\mathbf{u}}$ and $\mathrm{var}(a(\mathbf{x})^T \mathbf{u}) = \left\| \Sigma^{1/2} a(\mathbf{x}) \right\|_2^2$, equation above can be formulated as:

$$\sup_{\mathbf{u}\sim\mathbb{Q}(\bar{\mathbf{u}},\mathbf{\Sigma})}\left\{\text{Prob}\left(\frac{a(\mathbf{x})^T\mathbf{u}-a(\mathbf{x})^T\bar{\mathbf{u}}}{\left\|\mathbf{\Sigma}^{1/2}a(\mathbf{x})\right\|_2}\geq\frac{-b(\mathbf{x})-a(\mathbf{x})^T\bar{\mathbf{u}}}{\left\|\mathbf{\Sigma}^{1/2}a(\mathbf{x})\right\|_2}\right)\right\}\leq\varepsilon \tag{12}$$

For a random variable z, Let's define a function $\Lambda(\rho):\mathbb{R}_+\rightarrow[0,1]$ as:

$$\Lambda(\rho)=\sup_{\mathbb{Q}(0,1)}\text{Prob}(z\geq\rho) \tag{13}$$

The inverse of this function (if applicable) is denoted $\Lambda^{-1}(\rho):[0,1]\rightarrow\mathbb{R}_+$.
Working with the random variable $a(\mathbf{x})^T\mathbf{u}$ and in the light of the above definition, we get:

$$\sup_{\mathbf{u}\sim\mathbb{Q}(\bar{\mathbf{u}},\mathbf{\Sigma})}\left\{\text{Prob}\left(\frac{a(\mathbf{x})^T\mathbf{u}-a(\mathbf{x})^T\bar{\mathbf{u}}}{\left\|\mathbf{\Sigma}^{1/2}a(\mathbf{x})\right\|_2}\geq\frac{-b(\mathbf{x})-a(\mathbf{x})^T\bar{\mathbf{u}}}{\left\|\mathbf{\Sigma}^{1/2}a(\mathbf{x})\right\|_2}\right)\right\}=\Lambda\left(\frac{-b(\mathbf{x})-a(\mathbf{x})^T\bar{\mathbf{u}}}{\left\|\mathbf{\Sigma}^{1/2}a(\mathbf{x})\right\|_2}\right) \tag{14}$$

As a result, equation (12) can be reformulated as:

$$\Lambda\left(\frac{-b(\mathbf{x})-a(\mathbf{x})^T\bar{\mathbf{u}}}{\left\|\mathbf{\Sigma}^{1/2}a(\mathbf{x})\right\|_2}\right)\leq\varepsilon \tag{15}$$

Or,

$$\frac{-b(\mathbf{x})-a(\mathbf{x})^T\bar{\mathbf{u}}}{\left\|\mathbf{\Sigma}^{1/2}a(\mathbf{x})\right\|_2}\leq\Lambda^{-1}(\varepsilon) \tag{16}$$

With a re-arrangement, we get:

$$a(\mathbf{x})^T\bar{\mathbf{u}}+\Lambda^{-1}(\varepsilon)\left\|\mathbf{\Sigma}^{1/2}a(\mathbf{x})\right\|_2+b(\mathbf{x})\geq 0 \tag{17}$$

which is also a deterministic constraint. It is readily seen that the constraint in equation (17), emanating from a linear stochastic constraint, has the exact same form as equation (8), which was originated from a linear robust constraint. For the entire equivalency, we should have:

$$\mathbf{u}_0=\bar{\mathbf{u}} \tag{18}$$

And,

$$\mathbf{P}=\gamma\mathbf{\Sigma}^{1/2} \tag{19}$$

Where,

$$\gamma=\Lambda^{-1}(\varepsilon) \tag{20}$$

Equation (18) indicates that for a robust constraint described by an ellipsoidal set (see equation (5)) to be equivalent to a probabilistic constraint (as in equation (10)), its center should be placed at the mean point $\bar{\mathbf{u}}$ of the uncertain parameters. In addition, the shape matrix $\mathbf{P}$ of the ellipsoidal set is related to the covariance matrix by $\gamma\mathbf{\Sigma}^{1/2}$. The underlying uncertainty ellipsoid can therefore be written in different forms as below. Such an ellipsoid can be interpreted as an affine transformation of a ball of radius γ .

$$\Xi(\mathbf{u})=\left\{\mathbf{u}_0+\mathbf{P}t\mid\left\|t\right\|_2\leq 1\right\} \tag{21}$$

$$=\left\{\bar{\mathbf{u}}+\gamma\boldsymbol{\Sigma}^{1/2}t \mid \|t\|_2 \leq 1\right\}$$
$$=\left\{\bar{\mathbf{u}}+\boldsymbol{\Sigma}^{1/2}t \mid \|t\|_2 \leq \gamma\right\}$$
$$=\left\{(\mathbf{u}-\bar{\mathbf{u}})^T\boldsymbol{\Sigma}^{-1}(\mathbf{u}-\bar{\mathbf{u}}) \leq \gamma^2\right\}$$

Inversely, if we start from a robust linear constraint as in equation (5), it is possible to define a probabilistic counterpart for it having the form as in equation (9), via using (18) and (19).
The main question that remains is how to calculate the ellipsoidal size factor γ, or the value of the function Λ and its inverse (see equation (20)).
If the classical one-sided Chebychev inequality (after Cantelli [13]) is used, the result by Calafiore and Ghaoui [14] indicates that $\gamma = ((1-\varepsilon)/\varepsilon)^{0.5}$. This means that in order to obtain a maximum target failure probability of ε, under the assumption of the first two moments being known, one can use an ellipsoidal set with this size factor γ and perform a robust optimization. Such a set is the largest possible ellipsoidal set that safely warrants a maximum target failure probability of ε. Inversely, if an ellipsoidal set with such a size factor γ is used for robust (or non-probabilistic) design of a structure, it is as if we are guaranteeing a minimum failure probability of $1/(1+\gamma^2)$. Such a minimum failure probability is guaranteed, with any realization of the underlying probability distribution for $\mathbf{u}$, having the (true) given means and covariances.
Through using the Chebychev inequality, the criticism of pushing the limits of reliability-based design (and structural reliability analysis) by grossly assuming particular distribution types is to a good extent eliminated. However, using this classical inequality, even though it provides a tight bound, results in a worst-case distribution which will be discrete and can have few atoms. In other words, the ellipsoidal uncertainty set might be too large (large γ), or equivalently the guaranteed failure probability can be quite weak (small ε). In spite of all this, it is emphasized that this notion has been extensively used across various disciplines (e.g. control applications) owing to the safe results it can provide. It might similarly be preferred in structural and mechanical engineering applications where the underlying data and knowledge is very limited and the stakeholder is not at comfort to make assumptions regarding the distribution types.
A mild additional assumption that can be imposed on the structure of the worst probability distribution (belonging to the ambiguous family $\mathbb{Q}(\bar{\mathbf{u}},\Sigma)$) is unimodality. Crudely speaking, a univariate continuous probability density function is unimodal if it is non-increasing as we get away from a central point called the *mode*. In other words, smaller probability densities are assigned as the deviation from mode increases, a property that is reasonable to assume in many practical applications. Several of the commonly used parametric probability distributions are unsurprisingly unimodal. Few examples are normal, Cauchy, Student, chi-square, logistic, beta (α, β>1), and gamma (α, β>1). Unimodality is a critical piece of additional information, beyond the knowledge of first and second moments, that can result in much less conservative results relative to the scenario where the one-sided Chebychev inequality is used.
Gauss inequality [15] is a well-known subset of the Chebychev inequality given the property of unimodality for the distribution. From [15], it can be seen that the two-sided Chebychev bound for univariate random variables is subject to an improvement by a considerable factor of $4/9$ if the distribution is unimodal. As equation (13) indicates, we are interested in a one-sided inequality (rather than a two-sided one), by which we can calculate the supremum probability of a random variable exceeding a certain limit (this limit is often described in the unit of standard deviations). The one-sided version of Gauss inequality does not exist analytically (as it does for the Chebychev inequality after Cantelli [13]). However, it can be numerically calculated using the recent results by [16] through solving a semidefinite program (SDP) with

matrix inequalities. The proposed methodology in the mentioned reference can be more generally used to calculate the probability of falling outside any arbitrary polytope (a closed convex set with linear boundaries) in the uncertain parameter space given the assumption of first and second moments in addition to unimodality. In this article, we cover a single probabilistic constraint, rather than multiple ones. The unimodal family of distributions is denoted by $\mathbb{Q}_\alpha$.

One-Sided Univariate Gauss Inequality [16]. The upper bound for $\mathrm{Prob}(u-\overline{u}>\rho)$ when u belongs to the family of unimodal probability distributions $\mathbb{Q}_\alpha(\overline{u},\sigma)$ with known first and second moments is the solution to the following optimization problem.

$$
\begin{aligned}
\sup_{\mathbb{Q}_\alpha(\mu,\sigma^2)} \ \mathrm{Prob}(u-\overline{u}\geq\rho) &= \text{maximize } \lambda - t_0 \\
\text{subject to} \quad & z\in\mathbb{R},\ Z\in\mathbb{R}_+,\ \lambda\in\mathbb{R},\ t\in\mathbb{R}^{l+1} \\
& \begin{bmatrix} Z & z \\ z & \lambda \end{bmatrix} \succeq 0,\ z\geq 0,\ t\geq 0 \\
& \begin{bmatrix} Z & z \\ z & \lambda \end{bmatrix} \preceq \begin{bmatrix} \frac{\alpha+2}{\alpha}\sigma^2 & \frac{\alpha+1}{\alpha}\mu \\ \frac{\alpha+1}{\alpha}\mu^T & 1 \end{bmatrix} \\
& \left\| \begin{bmatrix} 2\lambda\rho \\ t_l\rho - z \end{bmatrix} \right\|_2 \leq t_l\rho + z \\
& \left\| \begin{bmatrix} 2t_{j+1} \\ t_j - \lambda \end{bmatrix} \right\|_2 \leq t_j + \lambda \quad \forall j\in E \\
& \left\| \begin{bmatrix} 2t_{j+1} \\ t_j - \lambda \end{bmatrix} \right\|_2 \leq t_j + t_l \quad \forall j\in O
\end{aligned}
\tag{22}
$$

where,

$$
\begin{aligned}
l &= \left\lceil \log_2 \alpha \right\rceil, \\
E &= \left\{ j\in\{0,\dots,l-1\} :\ \left\lceil \alpha / 2^j \right\rceil \text{ is even} \right\}, \\
O &= \left\{ j\in\{0,\dots,l-1\} :\ \left\lceil \alpha / 2^j \right\rceil \text{ is odd} \right\}
\end{aligned}
$$

Function $\Lambda(\rho)$ (see equation (13)) can therefore in this case be evaluated by setting $\mu=0$ in the optimization problem above and solving it. A subscript can be added to the function symbol (Λ_α) to indicate the value of α that is used.
Parameter α in equations above is in line with the definition of the concept of $\alpha-$unimodality, which is a generalized notion of unimodality. In theory, the parameter α can vary in the range $(0,\infty)$. For the above univariate case, if α is set to 1 in (22), where 1 is in fact the dimension of the uncertainty vector, we get the one-sided version of the celebrated two-sided Gauss inequality as in [15]. Interestingly, if α approaches infinity in this case, it can be shown that solving problem (22) results in the one-sided Chebychev bound after Cantelli [13]. Our previous notation $\mathbb{Q}(\overline{u},\sigma)$ for the Chebychev-based family of distributions can therefore be more precisely written as $\mathbb{Q}_\infty(\overline{u},\sigma)$. If α approaches 0 (which is not of particular interest to us), the extreme probability density tends to the Dirac delta function at the mode. The two practical and readily interpretable α values for univariate cases are 1 and infinity, as just explained.

For multivariate cases, a natural supposition for the parameter α is $\alpha = nu$ which is the dimension of the uncertainty vector $\mathbf{u} \in \mathbb{R}^{nu}$. This assumption creates a so-called star-unimodal distribution. Star-unimodal distributions have a non-increasing density along any ray emanating from their mode (often taken as $\mathbf{0}$ without loss of generality), which is an intuitive analogue of the univariate definition of unimodality. A multivariate normal density is for instance star-unimodal. However, star-unimodality can generate much more diverse and noninnocent-looking distributions. The important class of star-unimodal distributions $\mathbb{Q}_{\alpha=nu}(\bar{\mathbf{u}}, \mathbf{\Sigma})$ is simply denoted $\mathbb{Q}_*(\bar{\mathbf{u}}, \mathbf{\Sigma})$.
We do not plan to get into more mathematical details regarding the notion of $\alpha -$unimodality in order to limit the length of this paper. We refer the interested reader to [17] and [16] for additional details.
So far, we treated the case where $\mathbf{u}$ belonged to an ambiguous family of distributions $\mathbb{Q}_\infty(\bar{\mathbf{u}}, \mathbf{\Sigma})$ or $\mathbb{Q}_\alpha(\bar{\mathbf{u}}, \mathbf{\Sigma})$ (and its subset $\mathbb{Q}_*(\bar{\mathbf{u}}, \mathbf{\Sigma})$). We now treat the case where $\mathbf{u}$ has a known distribution. The most common case is when $\mathbf{u}$ is normally distributed $\mathbf{u} \sim N(\bar{\mathbf{u}}, \mathbf{\Sigma})$. This situation is also consistent with the principle of maximum entropy. For a continuous distribution with a given variance and an arbitrary (but known) mean, it can be shown that the distribution with maximum entropy is normal. This situation with normal $\mathbf{u}$ (either known, or based on the maximum entropy principle) can be treated as a subcategory of the previous cases that were just discussed.
We take the discussions that were given previously in this section from equation (12) onwards. The supremum in this equation needs to be dropped as we are dealing with a known distribution, rather than an ambiguous one. We will therefore have:

$$\mathrm{Prob}\left(\frac{a(\mathbf{x})^T \mathbf{u} - a(\mathbf{x})^T \bar{\mathbf{u}}}{\left\| \Sigma^{1/2} a(\mathbf{x}) \right\|_2} \geq \frac{-b(\mathbf{x}) - a(\mathbf{x})^T \bar{\mathbf{u}}}{\left\| \Sigma^{1/2} a(\mathbf{x}) \right\|_2} \right) = 1 - \Phi\left(\frac{-b(\mathbf{x}) - a(\mathbf{x})^T \bar{\mathbf{u}}}{\left\| \Sigma^{1/2} a(\mathbf{x}) \right\|_2} \right) \leq \varepsilon \tag{23}$$

where Φ is the cumulative density function (CDF) of the standard normal distribution. This is simply because $\mathbf{a}(x)^T \mathbf{u}$ is normally distributed, which is standardized by deducting its mean and dividing by its standard deviation inside the probability operator in equation above.
In other words, function $\Lambda(\rho)$ can in this case be related to the CDF of the standard normal distribution by:

$$\Lambda(\rho) = \mathrm{Prob}(z \geq \rho) = 1 - \Phi(\rho) = \Phi(-\rho) \tag{24}$$

The function $\Lambda(\rho)$ can be in general (for the Chebychev and Gauss inequality cases) be considered as a cumulative belief function, considering a Depmster-Shafer structure. This function, as shown above, degenerates to a precise cumulative distribution function when the uncertainty is fully described. Further discussion about which is left out of the scope of this article.
Using equation (23), we get:

$$\Phi\left(\frac{-b(\mathbf{x}) - a(\mathbf{x})^T \bar{\mathbf{u}}}{\left\| \Sigma^{1/2} a(\mathbf{x}) \right\|_2} \right) \geq 1 - \varepsilon \tag{25}$$

By applying the inverse operator Φ^{-1} on both sides of equation above and a rearrangement, we finally get:

$$a(\mathbf{x})^T \bar{\mathbf{u}} + \Phi^{-1}(1 - \varepsilon) \left\| \Sigma^{1/2} a(\mathbf{x}) \right\|_2 + b(\mathbf{x}) \geq 0 \tag{26}$$

which is also a deterministic constraint, having the exact same form as before. The ellipsoidal size factor in this case is $\gamma = \Phi^{-1}(1-\varepsilon) = \Phi^{-1}(\eta) = -\Phi^{-1}(\varepsilon)$, which with no accident coincides with the celebrated reliability index in structural reliability literature. As a result, every linear probabilistic constraint with a normal description has an ellipsoidal *robust counterpart*. The opposite also applies. Any linear robust constraint whose parameters are described by an ellipsoidal set can be interpreted as a Gaussian probabilistic constraint. In other words, probabilistic and robust optimization problems (1) and (2) can be viewed as exactly equivalent in this case.

Calafiore and Ghaoui [14] showed that this conclusion applies to probabilistic constraints with parameters described by any radial distribution. Another interesting case (beyond normal distribution) is the uniform distribution on ellipsoidal support, which is radial and hence enjoys from having a robust counterpart-even though radiality is not a necessary requirement for having a robust counterpart.

Let us now introduce the transformation $\mathbf{z} = T(\mathbf{u}) = \mathrm{Q}\,\mathbf{u} + \mathbf{q}$, defined by $\mathrm{Q} = \mathbf{\Sigma}^{1/2}$ and $\mathbf{q} = -\mathbf{\Sigma}^{1/2}\overline{\mathbf{u}}$. Such a transformation, applied on a normal uncertain vector $\mathbf{u}$ is equivalent to its transformation into an uncorrelated and standardized space. In other words, the transformation converts $\mathbf{u} = N(\overline{\mathbf{u}}, \Sigma)$ into $\mathbf{z} = N(\mathbf{0}, \mathbf{I})$. Additionally, it can be easily shown that $T(\mathbf{u})$ transforms the uncertainty ellipsoid in equation (21) to a ball of radius γ centered at $\mathbf{0}$, as formulated below:

$$\mathrm{B}(\mathbf{z}) = \left\{ (\mathbf{z} - \overline{\mathbf{z}})^T (\mathbf{z} - \overline{\mathbf{z}}) \leq \gamma^2 \right\} \tag{27}$$

In this new space, both the stochastic and robust constrains can be converted into the commonly presented deterministic constraint:

$$\gamma \left\| a(\mathbf{x}) \right\|_2 + b(\mathbf{x}) \leq 0 \tag{28}$$

with γ remaining the same as previously discussed. This simply indicates that for our particular problem, instead of working in space $\mathbf{u}$ of uncertain parameters, we could choose to work in the transformed space $\mathbf{z}$ with no difference. This transformation is not necessary from computational point of view, when normally-distributed vectors are involved (essentially even with nonlinear failure surfaces, as will be discussed in the next section). However, it may prove conceptually useful because a ball has a simpler description than an ellipsoid. In addition, we will be dealing with a probabilistic space where advancement in any direction is equivalent, which is conceptually appealing. However, this is the well-known key to treatment of non-Gaussian variables in advanced FORM/SORM analysis (and FORM/SORM based reliability-based optimization). Through using the renowned Rosenblatt transform (or other transforms), the uncertain parameters $\mathbf{u}$ with any joint probability distribution $p_{\mathbf{u}}(\mathbf{u})$ can be transformed into an uncorrelated standardized space $\mathbf{z}$: $\mathbf{z} = R(\mathbf{u})$. With this, we are in a similar $\mathbf{z}$ environment as just described. Even without any transformation into $\mathbf{z}$ space, the robust design is still meaningful in original space. The shape of the uncertainty set is however not anymore ellipsoidal (or ball-like) but rather decided by the form of the underlying probabilistic model i.e. the joint probability distribution $p_{\mathbf{u}}(\mathbf{u})$. The shape of the bounded uncertainty sets can in fact be taken as the shape of the iso-density contours of the joint probability distribution. However, parametrizing the shape of the uncertainty sets (using a single parameter like γ) becomes more challenging, and the calculation of such a size factor based on a target reliability gets more demanding. We do not plan to elaborate further in this direction.

3 ON THE NONLINEARITY OF THE FAILURE SURFACE

We first give two definitions.

Definition: Quasiconvexity and Quasiconcavity [12]. A function $f: \mathbb{R}^n \to \mathbb{R}$ is *quasiconvex* if its domain and all its sublevel sets are convex. The sublevel sets are defined as: $S_\alpha = \left\{ u \in \mathbf{dom}\ f \mid f(u) \le \alpha \right\}$ for $\alpha \in \mathbb{R}$. A function is *quasiconcave* if $-f$ is quasiconvex. In other words, if its superlevel sets $S_\alpha = \left\{ u \in \mathbf{dom}\ f \mid f(u) \ge \alpha \right\}$ are convex.

Quasiconvexity is a significant generalization of convexity. This is because it deals with convexity of the lowerlevel sets of a function rather than the function itself. It is easy to show that all quasiconvex functions are convex, but that the opposite obviously does not hold. The same applies to quasiconcavity.

Let us get back to our safety indicator function $h(\mathbf{x}, \mathbf{u})$. According to our definition, we know that the superlevel sets of this function with respect to $\mathbf{u}$, $S_0 = \left\{ \mathbf{u} \in \mathbf{dom}\ h \mid h(\mathbf{u}) \ge 0 \right\}$, indicate failure. Therefore, the convexity of failure region is equivalent to quasiconcavity of h. As $h = -g$, this is equal to quasiconvexity of the safety margin g.

Various quasiconvex safety margins $g(\tilde{\mathbf{x}}, \mathbf{u}) : \mathbb{R}^2 \to \mathbb{R}$ (or quasiconcave h functions) are shown in Figure 1 via illustrating their level sets in the $\mathbf{u} \in \mathbb{R}^2$ space. The failure curves $g = 0$ are shown by bold boundaries, all touching an uncertainty ellipsoid. The dark gray regions are the sublevel sets of the g functions (i.e. $g \le 0$) or the failure region which are all convex sets. Linear, quadratic, and polyhedral boundaries among other types can all mark convex failure regions and therefore quasiconvex safety margins. This would of course depend on the side the sublevel sets of g fall upon, or we will inversely get a quasiconcave safety margin. The illustrations are given in conjunction with a robust design optimization. Apparently, the same applies to a probabilistic design optimization.

Quasiconvexity of safety margin g has an immediate implication in our design optimization problems. This property ensures that the actual failure region remains inside an approximate linearly-bounded failure region, as shown in Figure 2. A robust-optimal design is shown in this figure with optimal values $\mathbf{x}^*$ for the design variables. The safety margin is therefore projected as $g(\mathbf{x}^*, \mathbf{u}) = g(\mathbf{u})$ in the uncertain parameter space. Working with the linear boundary in order to estimate the failure probability guarantee of a robust design optimization results in safe approximations in this case. However, if the safety margin becomes quasiconcave, one can be certain that the linear approximation results in under-designs.

In scenarios that we neither have a quasiconvex nor a quasiconcave safety margin, one cannot be certain whether an under- or an over-design is resulted. However, we still have an approximate result in place. In these scenarios (also with quasiconcave safety margins), the probability of falling outside the entire uncertainty ellipsoid can be used as a safe approximation. The other alternative would be to evaluate the exact failure probability by Monte Carlo simulation (MCS) when the probability distributions are known (which is evident). In the case of Chebychev-based ambiguity set $\mathbb{Q}_\infty(\overline{\mathbf{u}}, \mathbf{\Sigma})$, the results by [18] can be used when failure surfaces are quadratic (or can be quadratically-approximated). Using the results in [18], we can deal with the lower-level optimization problem, to be embedded in the higher level problem of design optimization. In the case of Gauss-based ambiguity set $\mathbb{Q}_\alpha(\overline{\mathbf{u}}, \mathbf{\Sigma})$, the failure region (or its complement) can be safely approximated by a polytopic convex hull and the results after [16] can be applied. To limit the length of this paper, we do not further explain in this direction.

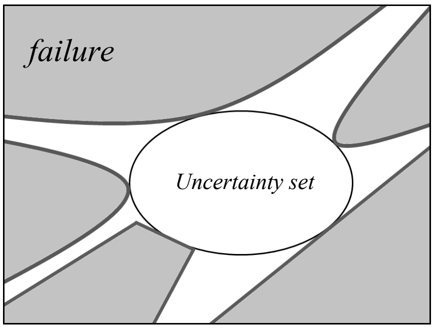

Figure 1 Five instances of quasiconvex safety margin g (equivalent to quasiconcave h) in a robust design optimization.

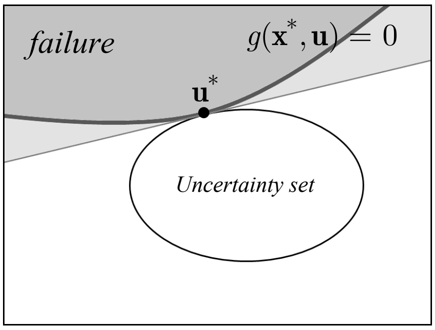

Figure 2 A quasiconvex safety margin and its linear safe approximation in a robust design optimization.

4 SUMMARY OF RESULTS

In this section, we summarize the results of Section 2, in the light of the nonlinearity discussions that were given in Section 3. A comparison is given in Table 1. In this table, the equivalent probabilistic and robust linear constraints are presented for different assumptions related to the probability distributions. Such an equivalence is a two-way relationship, and is exact for linear (with respect to $\mathbf{u}$) safety margins.

Probabilistic Constraint	Equivalent Robust Constraint
$\text{Prob}(\mathbf{a}^T\mathbf{u} + b \leq 0) \geq 1 - \varepsilon$ $\mathbf{u} \sim N(\bar{\mathbf{u}}, \mathbf{\Sigma})$ Maximum Entropy Distribution	$\mathbf{a}^T\mathbf{u} + b \leq 0$ $\forall \mathbf{u} \in \Xi(\mathbf{u}) = \left\{ \bar{\mathbf{u}} - \phi^{-1}(\varepsilon)\Sigma^{1/2}t \mid \left\| t \right\|_2 \leq 1 \right\}$
$\text{Prob}(\mathbf{a}^T\mathbf{u} + b \leq 0) \geq 1 - \varepsilon$ $\mathbf{u} \sim \mathbb{Q}_\alpha(\bar{\mathbf{u}}, \mathbf{\Sigma})$ Gauss-based Family of Distribution	$\mathbf{a}^T\mathbf{u} + b \leq 0$ $\forall \mathbf{u} \in \Xi(\mathbf{u}) = \left\{ \bar{\mathbf{u}} + \Lambda_\alpha^{-1}(\varepsilon)\Sigma^{1/2}t \mid \left\| t \right\|_2 \leq 1 \right\}$
$\text{Prob}(\mathbf{a}^T\mathbf{u} + b \leq 0) \geq 1 - \varepsilon$ $\mathbf{u} \sim \mathbb{Q}_\infty(\bar{\mathbf{u}}, \mathbf{\Sigma})$ Chebychev-based Family of Distribution	$\mathbf{a}^T\mathbf{u} + b \leq 0$ $\forall \mathbf{u} \in \Xi(\mathbf{u}) = \left\{ \bar{\mathbf{u}} + ((1-\varepsilon)\ /\ \varepsilon)^{0.5}\Sigma^{1/2}t \mid \left\| t \right\|_2 \leq 1 \right\}$

Table 1 Summary of equivalent probabilistic and robust constraints based on different assumptions for the underlying probability distribution.

If a robust optimization with an ellipsoidal set of size γ is performed, it is as if we have satisfied the failure probabilities listed in Table 2 depending on the strength of assumption about the probability distribution of the underlying data given its first and second moments. Such a failure probability is guaranteed for failure surfaces that are linear with respect to uncertain parameters $\mathbf{u}$, or in the more general case where $h(\mathbf{x},\mathbf{u}) = -g(\mathbf{x},\mathbf{u})$ is quasiconcave in $\mathbf{u}$. In the linear case, the failure probability is sharply guaranteed. However, in the case of quasiconcave $h(\mathbf{x},\mathbf{u})$, the probability of failure is not guaranteed sharply. This means that one can be certain that the failure probability cannot be larger than the listed probability guarantee, but might be smaller (which is acceptable). In other circumstance i.e. when $h(\mathbf{x},\mathbf{u})$ is not quasiconcave, the given failure probability is only an approximate failure probability guarantee, meaning that it could be exceeded. In this situation, one can consult with the discussions given at the end of Section 3.

In an inverse fashion, with a target failure probability of ε in mind, the guaranteed size of the uncertainty set can be determined as presented in Table 3. In other words, if the ellipsoidal set is defined by the proposed size factors, we can be certain that the given target failure probability is achieved. The size of the ellipsoidal set is exact for linear safety margins and larger than what it could have been if $h(\mathbf{x},\mathbf{u}) = -g(\mathbf{x},\mathbf{u})$ was quasiconcave. In any case, the ellipsoidal set has a guaranteed size as it results in an acceptable design (which may be marginally conservative if $h(\mathbf{x},\mathbf{u})$ becomes quasiconcave). If $h(\mathbf{x},\mathbf{u})$ is not quasiconcave, the design ellipsoid given in Table 3 can produce (in many practical cases only marginally) non-conservative designs. This non-conservativeness can be removed by following the discussions briefly presented at the end of Section 3.

Assumption	Failure Probability Guarantee (Given γ)
Maximum Entropy Distribution	$\Phi(-\gamma)$
Gauss-based Family of Distribution	$\Lambda_\alpha(\gamma)$
Chebychev-based Family of Distribution	$1 \,/\, (1+\gamma^2)$

Table 2 Summary of failure probability guarantee given the size of the uncertainty set γ .

Assumption	Guaranteed ellipsoidal size factor (Given ε)
Maximum Entropy Distribution	$-\Phi^{-1}(\varepsilon)$
Gauss-based Family of Distribution	$\Lambda_\alpha^{-1}(\varepsilon)$
Chebychev-based Family of Distribution	$((1-\varepsilon) \,/\, \varepsilon)^{0.5}$

Table 3 Summary of the guaranteed size of the uncertainty set given the target failure probability ε .

5 NUMERICAL EXAMPLE

Consider the simply-supported on-way concrete roof slab illustrated in Figure 3 that is reinforced in the lower (tensile) part of the cross-section. The problem is taken from [1], Example 1.1., and modified. The slab has the span length L (determined by architectural considerations), the height h (determined by experience, pre-engineering and construction restraints which is often a direct function of L), and an effective height h_e (often characterizable as a function of beam height and therefore the span length in practice; This can be verified after calculating the neutral axis based on compressive part of concrete). The main designer's choice with a predefined span length will therefore be the reinforcement cross-sectional area A_s. The reinforcement steel strength f_s is often pre-decided based on market situation and common practice. It is therefore assumed here that a safe deterministic estimate of f_s is available to the designer. In other words, f_s is treated as a deterministic design parameter. This assumption has been made as sufficient data exists about steel manufacturing subject to strict quality controls, especially in comparison with the loads exerted on the beam which are considered the main source of uncertainty. The beam is subjected to dead load (material weight), live load (human activity load) and snow (environmental) loads at the top. All the loads are treated quasi-statically i.e. no dynamics involved. The main uncertainty is attributed to the live (u_1) and snow (u_2) loads, as we have deficient knowledge and statistical data about them. The dead load (w) is treated deterministically by using its safe conservative estimate based on our existing knowledge and experience. The failure in this example is determined by formation of a plastic hinge in mid-span as a result of the external loads. The bending moment capacity of the slab is $M_p = f_s h_e A_s$ (ignoring the effect of concrete compressive strength) and the mid-span bending moment is $M_m = 0.125L^2(\mathrm{w} + u_1 + u_2)$. The safety margin is therefore formulated as:

$$\begin{aligned} g(\mathbf{x},\mathbf{u}) &= M_p - M_m \\ &= f_s h_e A_s - (0.125L^2 u_1 + 0.125L^2 u_2 + 0.125L^2 w) \\ &= f_s \underbrace{(h_e A_s \ / \ L^2)}_{x_1} - (0.125u_1 + 0.125u_2 + 0.125w) \\ &= f_s x_1 - 0.125u_1 - 0.125u_2 - 0.125w \end{aligned} \tag{29}$$

In this formulation, the design variable is taken as $x_1 = h_e A_s \ / \ L^2$ resulting in a biaffine (affine with respect to both $\mathbf{x}$ and $\mathbf{u}$) safety margin. The ratio $h_e \ / \ L^2$ is often a constant for practical ranges of span length. The major design variable A_s can therefore be simply recovered from x_1. Biaffinity of the safety margin is not a must, as we discussed in the article, but simplifies the situation by creating a tractable optimization problem (i.e. a second-order cone program or an SOCP). Otherwise, we would have to resort to methods such as sequential programming.

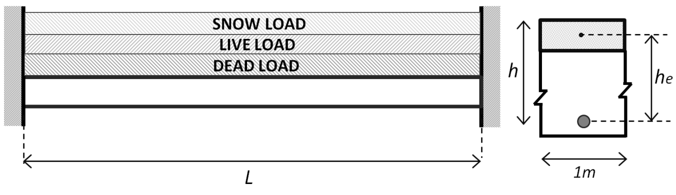

Figure 3 A reinforced concrete roof slab.

Based on the existing historical data, the annual maximum values of the live and snow loads (for a particular type of building and geographical region) are determined to have the expected values: $\bar{\mathbf{u}} = [30 \quad 37]^T \mathrm{kN/m^2}$ and the covariances $\mathbf{\Sigma} = \left[(30 \times 0.1)^2 \quad 0 \quad ; \quad 0 \quad (37 \times 0.25)^2\right](\mathrm{kPa})^2$. The semicolon symbol is used to separate the matrix rows. The stochastic formulation of the problem is given below. The objective function is defined as $f(\mathbf{x}) = x_1$, resulting in the amount of reinforcement A_s to be minimized (given a constant $h_e \ / \ L^2$). Given the uncertainty in the loads, the purpose of the design would be to determine the amount of reinforcement such that the safety constraint $g(\mathbf{x},\mathbf{u}) \ \geq 0$ is satisfied with a specified annual probability. For this simple numerical exercise, we require that the annual failure probability is at most 0.001. The problem is formulated as:

$$\begin{aligned} &\underset{\mathbf{x}=x_1}{\text{minimize}} \;\; f(\mathbf{x}) = x_1 \\ &\text{subject to} \quad \text{Prob}(\underbrace{g(\mathbf{x},\mathbf{u}) \;\; \geq 0}_{Safe\ State}) \geq \underbrace{1-0.001}_{0.999} \end{aligned} \tag{30}$$

Which as shown in the paper has an exact robust counterpart for Normal, Gauss-based and Chebychev-based family of distributions for $\mathbf{u}$:

$$\begin{aligned} &\underset{\mathbf{x}}{\text{minimize}} \;\; x_1 \\ &\text{subject to } g(\mathbf{x},\mathbf{u}) \;\; \geq 0 \;\; \forall \mathbf{u} \in \Xi(\mathbf{u}) = \left\{ (\mathbf{u}-\bar{\mathbf{u}})^T \boldsymbol{\Sigma}^{-1} (\mathbf{u}-\bar{\mathbf{u}}) \leq \gamma^2 \right\} \end{aligned} \tag{31}$$

where the ellipsoidal size γ depends on the degree of assumption about the distribution on $\mathbf{u}$ according to Table 3. To create tangible results, we treat a realistic slab with $L = 4.0\text{m}$, $h = 0.4\text{m}$, $h_e = 0.75 \times h = 0.3\text{m}$, $f_s = f_{ys} / 1.15 = 500 / 1.15 = 435\text{MPa}$. The design optimization is performed through solving the SOCPs corresponding to the robust formulations above. The size factor corresponding to the Gauss-based family of distributions is calculated using a MATLAB-based SDP solver according to equation (22). The results are listed in Table 4 below.

The design optimization is also performed (probabilistically) for the situation where $\mathbf{u}$ is lognormally or extreme-value distributed. Parameters of the extreme value distributions are determined based on the original data as $u_1 \sim ev(3.14, 0.23)$ and $u_2 \sim ev(4.12, 0.72)$, which are of course consistent with the data means and covariances as given before. Both these cases are handled by FORM, as well as Monte Carlo simulation. The presented required reinforcements are based on MCS, as it is slightly more precise.

	Live Load u_1 / Snow Load u_2	Size Factor γ	Required Reinforcement	
			$A_s (m^2 / 1m)$	$\text{N}\phi\text{D}mm / 3m$
Fully Known Distribution	$\mathbf{u} \sim N(\bar{\mathbf{u}}, \boldsymbol{\Sigma})$	$-\Phi^{-1}(\varepsilon) = 3.1$	0.000304	$\cong 8\phi 12mm$
	$\mathbf{u} \sim evd(\mathbf{a}, \mathbf{b})$ with $\bar{\mathbf{u}}, \boldsymbol{\Sigma}$	3.03 (FORM)	0.000281 (MCS)	$\cong 8\phi 12mm$
	$\mathbf{u} \sim logN(\bar{\mathbf{u}}, \boldsymbol{\Sigma})$	3.09 (FORM)	0.000312 (MCS)	$\cong 9\phi 12mm$
Partially Known Distribution	$\mathbf{u} \sim \mathbb{Q}_*(\bar{\mathbf{u}}, \boldsymbol{\Sigma})$	$\Lambda_\alpha^{-1}(\varepsilon) = 22.3$	0.000661	$\cong 11\phi 15mm$
	$\mathbf{u} \sim \mathbb{Q}_\infty(\bar{\mathbf{u}}, \boldsymbol{\Sigma})$	$((1-\varepsilon) / \varepsilon)^{0.5} = 31.6$	0.000833	$\cong 14\phi 15mm$

Table 4 Summary of results.

The amount of reinforcement for all cases is presented per meter width of the one-way slab. Assuming that the slab has a width of 3 meters, the amount of steel is converted into the number of rebars of specified diameter. The Gauss-based family of distributions results in 11 rebars of diameter 15 in the width of the beam, as compared to the case corresponding to lognormal distribution which requires 9 rebars of diameter 12 in the same width. The increased number and diameter of the rebars is at the cost of not assuming any specific probability distribution for annual live and snow load data, except that their joint distribution is unimodal. If the target reliability was smaller, the difference between results of the ambiguous and parametric distributions would be reduced.

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