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NATURAL FREQUENCIES OF A STRUCTURE WITH EPISTEMIC UNCERTAINTY

Hesheng Tang[1], Dawei Li[2], Wen Yao[2], and Songtao Xue[2]

[1] State Key Laboratory for Disaster Reduction in Civil Engineering, Tongji University
Shanghai 200092, China
e-mail: thstj@tongji.edu.cn

[2] Research Institute of Structural Engineering and Disaster Reduction, Tongji University
Shanghai 200092, China
e-mail:lidaweicc123@163.com

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Abstract. *Uncertainties associated with eigenvalue problems play an important role in the dynamic analysis of engineering systems, which are perhaps the most essential to determine system dynamic behaviors. Due to lack of knowledge or incomplete, inaccurate, unclear information in the modeling, there are limitations in using only one framework (probability theory) to quantify the uncertainty in the eigenvalue problem because of the impreciseness of data or knowledge. This study explores the use of evidence theory for frequency analysis of a structural system in the presence of epistemic uncertainty. The evidence theory is used to quantify the uncertainty present in the structure's parameters such as material properties. In order to alleviate the computational difficulties in the evidence theory based uncertainty quantification (UQ) analysis, a differential evolution based interval optimization for computing bounds method is developed. Numerical example problems that illustrate the developed algorithm with comparison to probability and interval eigenvalue solution are presented, and the computational efficiency and accuracy of this approach method are also investigated.*

1 INTRODUCTION

Uncertainties are unavoidable in the description of real-life engineering systems. When uncertainties are considered in structural dynamic problems, uncertainty quantification of structural natural frequencies play a crucial role as the dynamic response is governed by the frequencies. Uncertainties can be broadly divided into two categories depended on their natural: aleatory uncertainty and epistemic uncertainty [1]. Aleatory uncertainty is also referred to as irreducible uncertainty or inherent uncertainty due to the inherent variation associated with the physical system or the environment under consideration. Epistemic uncertainty, on the other hand, derives from lacking of knowledge or only grasping limit information in the modeling process. Several epistemic uncertainty quantification techniques have been explored, such as fuzzy set theory [2], possibility theory [3], interval analysis [4, 5, 6], evidence theory [7, 8] and random sets [9] etc. Among the above-mentioned methods, evidence theory has a much more flexible framework to quantify epistemic uncertainty from the perspective of its theoretical body. Evidence theory is widely used in uncertainty reasoning [10], pattern classification [11], data fusion [12], structural uncertainty analysis [13] and reliability analysis [14]. Nevertheless, evidence based uncertainty propagation involves computationally extremely demand due to uncertainty variable is represented by many discontinuous sets, instead of a smooth and continuous explicit probability density function in probability theory, which results in that evidence theory is still remains challenging for application in complex engineering problems though it has exhibited promising advantages in uncertainty modeling.

In this work, frequency analysis of a structural system with epistemic uncertainty is presented. Evidence theory is used to quantify the uncertainty present in the structure's parameters such as material properties. In order to alleviate the computational difficulties in the evidence theory based uncertainty quantification (UQ) analysis, a differential evolution based interval optimization for computing bounds method is developed. A typical example with aleatory and epistemic uncertainties is investigated to demonstrate accuracy and efficiency of the proposed method by comparing with interval algorithm and probability theory.

2 UNCERTAINTY QUANTIFICATION WITH EVIDENCE THEORY

2.1 Fundamentals of evidence theory

Evidence theory (DST) was first proposed by Dempster [7] and extended by Shafer [8]. With respect to a single measure in probability theory, evidence theory employs belief and plausibility measures to characterize uncertainty by indicating the confident degree to believe that event is true and not false, respectively. Similar with finite sample space in classical probability theory, frame of discernment (FD) is used in DST to denote the entire collection of mutually exclusive and exhaustive possible elementary propositions, and it is always represented by a symbol Ω. As a central concept of DST, basic belief assignment (BBA) is defined as a mapping from power set $2^{\Omega} \rightarrow [0, 1]$ to express the degree of belief in a proposition and should satisfy the following axioms:

$$\begin{cases} m(A) \geq 0 \\ m(\Phi) = 0 \end{cases} \tag{1}$$

$$\sum m(A) = 1 \text{ for each } A \subseteq \Omega \tag{2}$$

as long as $m(A)>0$, then subset A is named focal element. In the light of concept for belief and plausibility, the expression of these two measures for proposition B can be obtained from following:

$$Bel(B) = \sum_{A \subseteq B} m(A) \qquad \text{for all } B \subseteq \Omega \tag{3}$$

$$Pl(B) = \sum_{A \cap B \neq \phi} m(A) \qquad \text{for all } B \subseteq \Omega \tag{4}$$

where A represents different elements in 2^{Ω}. On account of the evocation of insufficient experimental data or empirical knowledge, the belief degree of event A cannot represent the confident degree of $\tilde{A}$, that is $Bel(A)+Bel(\tilde{A})\leq 1$, while $Pl(A)+Bel(\tilde{A})=1$, which are completely different from probability distribution function(PDF) in probability theory, that is $p(A)+p(\tilde{A})=1$. The expression of this relationship shows in Figure 1.

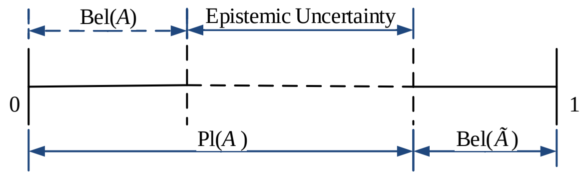


Figure 1: Uncertainty description of proposition.

In comparison with probability theory, evidence theory allows evidence stemming from different sources and employs the rules of combination to aggregate. One of most important combination rules is Dempster's rule which has following formulation:

$$m(B) = \frac{\sum_{A \cap C = B} m_1(A) m_2(C)}{1 - \sum_{A \cap C = \phi} m_1(A) m_2(C)} \qquad \text{for all } B \neq \phi \tag{5}$$

where $\sum_{A \cap C=\phi} m_1(A)m_2(C)$ can be viewed as contradict or conflict among the information given by the independent knowledge sources.

2.2 Uncertainty propagation with differential evolution

In the process of uncertainty propagation, the mathematical form of physical or finite element model can be abstractly expressed as:

$$\mathbf{y} = f(\mathbf{x}_v, \mathbf{d}) \tag{6}$$

where $\boldsymbol{y}= [y_1, y_2, \ldots, y_n]$ is the vector of system responses, and $\mathbf{x}_v=[x_1,x_2, \ldots,x_n]$ is the vector of uncertain input, $\mathbf{d}$ is the vector of deterministic input, f is transfer function. Just like in nature to the joint probability distribution density in probability theory, a joint belief structure shall be constructed to investigate the effects of uncertain variables. The consolidated joint belief structure can be vividly described as hypercube, because the uncertainty of each variable is represented by a set of intervals in evidence theory. From this prospective, the propagation of uncertainty can be viewed as a process to obtain the response bounds within each hypercube, corresponding mathematical formulation is described as:

$$\begin{array}{ll} \text{minimize} & f(x_i) \\ & \text{subject to } \underline{x_i} \le x_i \le \overline{x_i} \\ \text{maximize} & f(x_i) \end{array} \tag{7}$$

where $\underline{x_i}$ and $\overline{x_i}$ represent the lower and upper bounds of each joint hypercube. For the reason that the scale of variable is always large, in addition to every variable include many intervals, the uncertainty propagation process involved enormous joint hypercube is computationally extremely demand. Herein, a differential evolution (DE) strategy [15, 16, 17] is promoted to alleviate the burden of computation cost. As a novel evolutionary computation technique, differential evolution resembles the structure of an evolutionary algorithm (EA), but differs from traditional EAs in its generation of new candidate solutions and by its use of a 'greedy' selection scheme.

As a parallel direct search method, like other variant evolutionary algorithms, DE is initialized by randomly choosing *NP* population that cover the entire parameter space. Let $S \in$ Rn be the search space of the problem under consideration, the *n*-dimensional vector can be represented by $\boldsymbol{x}_i=(x_{i1}, x_{i2}, \ldots, x_{in})T \in$ S, *i*=1, 2, 3,…, *NP*. The DE algorithm is a population based algorithm like other genetic algorithm family involving the similar operators: crossover, mutation, crossover and selection.

(1)Mutation

The objective of mutation is to enable search diversity in the parameter space as well as to direct the existing object vectors with suitable amount of parameter variation in a way which will lead to better results at a suitable time. It keeps the search robust and explores new areas in the search domain.

According to the mutation operator, for each individual, $x_i^{(G)}$, *i*=1, 2, …, *NP*, at generation *G*, a mutation vector $v_i^{(G+1)} = \left(v_{i1}^{(G+1)}, v_{i2}^{(G+1)}, \cdots, v_{in}^{(G+1)}\right)^T$ is determined using different mutation methods. The mutation method of DE/current-to-best/1/bin recommended by Storn and Price [15] is used in this paper to optimize structure, which is corresponding to Eq.8:

$$v_i^{(G+1)} = x_i^{(G)} + F_1\left(x_{\text{best}}^{(G)} - x_i^{(G)}\right) + F\left(x_{r1}^{(G)} - x_{r2}^{(G)}\right) \tag{8}$$

where $x_{\text{best}}^{(G)}$=best individual of the population at generation *G*; *F* and F_1>0=real parameters, called mutation constants, which control the amplification of difference between two individuals so as to avoid search stagnation; and r_1, r_2 are mutually different integers, randomly selected from the set {1, 2,…, *i*-1, *i*+1,…, *NP*}.

(2) Crossover

Similar to genetic algorithms, following the mutation phase, the crossover operator is applied on the population. For each mutant vector, $v_i^{(G+1)}$ is a trial vector $u_{ij}^{(G+1)} = \left(u_{i1}^{(G+1)}, u_{i2}^{(G+1)}, \cdots, u_{in}^{(G+1)}\right)^T$ is generated, with

$$u_{ij}^{(G+1)} = \begin{cases} v_{ij}^{(G+1)} & \text{if}\left(\text{rand}(j) \le CR\right) or \left(j = \text{rand}n(i)\right) \\ x_{ij}^{(G+1)} & \text{if}\left(\text{rand}(j) > CR\right) or \left(j \ne \text{rand}n(i)\right) \end{cases} \tag{9}$$

In this equation, *j*=1, 2, …, *n*; rand(*j*) is the *j*th independent random number uniformly distributed in the range of [0, 1]. Randn(*i*) is a randomly chosen index from the set {1, 2,…, *n*}, and *CR* is user defined crossover constant $\in$[0, 1] that controls the diversity of the population.

(2) Selection

DE employs a greedy criterion which is different from genetic algorithms. After producing the offspring, the performance of the offspring vector and its parent is compared and the better one is selected. If the parent is still better, it is retained in the population, otherwise, the offspring retained in the population. The selection process is represented by Eq.10:

$$x_i^{(G+1)} = \begin{cases} u_i^{(G+1)} & \text{if} \left(f(u_i^{(G+1)}) < f\left(x_i^G\right)\right) \\ x_i^{(G+1)} & \text{otherwise} \end{cases} \tag{10}$$

To this end, the flow chart of differential evolution based quantification uncertainty is formulated as:

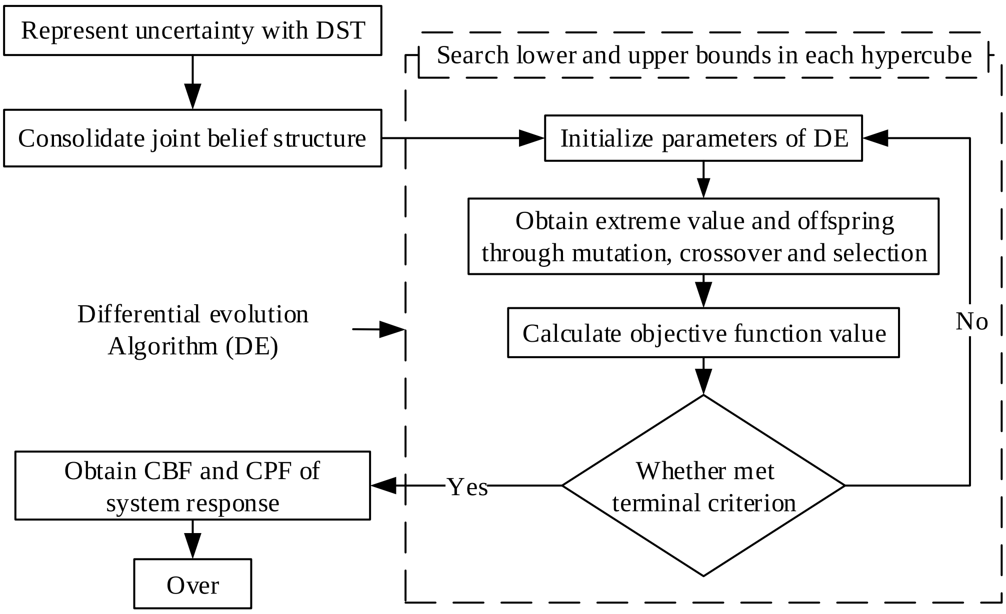


Figure 2: Flowchart of evidence theory for uncertainty quantification using differential evolution.

3 UNCERTAIN NATURAL FREQUENCIES ANALYSIS

3.1 Eigenvalue problem of structural vibration

The equilibrium equations for the free vibration of an undamped multi- degree of freedom system are defined as:

$$Mu'' + Ku = 0 \tag{11}$$

where M and K are the global mass and stiffness matrices of system, respectively; u is the displacement, u'' is the acceleration. The undamped eigenvalue problem is given by:

$$K\varphi = \omega^2 M\varphi \tag{12}$$

where ω, φ are respectively the circular frequencies and eigenvectors of system.

3.2 Uncertainty quantification of natural frequencies

In evidence theory, focal elements and BBA are determined firstly, and then focal elements between different variables combined to construct joint belief. It needs to find the maximum and minimum values of the natural frequencies respond ω to obtain the response bounds within each hyper-cube:

$$\begin{aligned} &\text{minimize} \quad \omega^2 = KM^{-1} \\ &\text{subject to} \quad \underline{E}_i \le E_i \le \overline{E}_i \end{aligned} \tag{13}$$

$$\begin{aligned} &\text{maximize} \quad \omega^2 = KM^{-1} \\ &\text{subject to} \quad \underline{E}_i \le E_i \le \overline{E}_i \end{aligned} \tag{14}$$

where ω represents the natural frequencies of system; $\underline{E}_i$ and $\overline{E}_i$ are the upper and lower bounds of modulus of elasticity E_i, respectively.

4 NUMERICAL EXAMPLE

An example problem solves the problem cited in the paper by [18, 19] using the interval method of the present work. The structure in the problem is a multi-spring-mass system with uncertainty in the elements' stiffness as shown in Figure 3. The element mass given in their work are $m_1 = m_2 = m_3 = m_4 = 1$. The uncertainties of stiffness described by multi-expert opinions in evidence theory are listed in Table 1. For comparison, the uncertainty of stiffness is also represented by probability theory and interval theory. Detailed information of probabilistic distribution(N(μ,σ)) and intervals for uncertain variables are listed in Table 2.

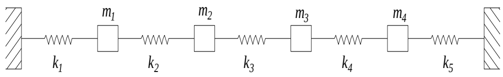


Figure 3: The system of multi-DOF spring-mass system.

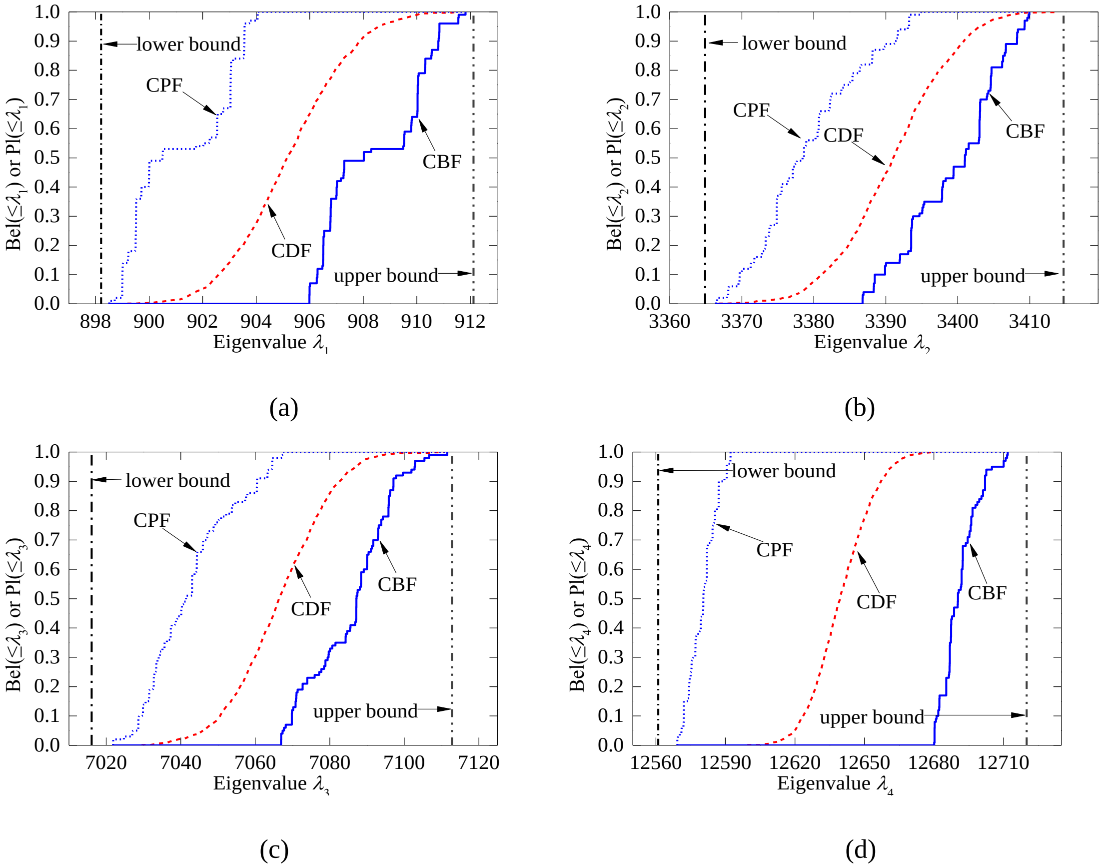
Variables		Focal element(N/m)	BBA
Expert 1	k_1	[990,1000][1000,1010]	0.55,0.45
	k_2	[1985,2015]	1
	k_3	[2980,3010][2990,3010][2990,3020]	0.2,0.6,0.2
	k_4	[3975,4025]	1
	k_5	[4970,5000][4980,5000][5000,5010][5010,5030]	0.1,0.4,0.3,0.2
Expert 2	k_1	[990,1010]	1
	k_2	[1985,1995][1995,2010][2010,2015]	0.35,0.4,0.25
	k_3	[2980,3020]	1
	k_4	[3975,4020][3975,4025]	0.8,0.2
	k_5	[4970,4990][4990,5010][5010,5030]	0.3,0.4,0.3

Table 1: Evidential representation for uncertain variables.

Variables	μ(N/m)	σ(N/m)	Interval(N/m)
k_1	999.5	4.5	[990,1010]
k_2	2000.625	8	[1985,2015]
k_3	3000	3	[2980,3020]
k_4	3998	1	[3975,4025]
k_5	5007.169	25	[4970,5030]

Table 2: Detailed information of probabilistic distribution and intervals for uncertain variables.

The eigenvalue problem is solved using the method presented in this work and results are plotted in Figure 4. The comparison of results obtained for the eigenvalue problem using the present method and the results obtained by probability and the interval methods are presented (Figure 4 and Table 3).



(a) (b)

(c) (d)

Figure 4: Eigenvalues cumulative distribution based on probability theory and evidence theory.

$\lambda_{(0.95)}$	Evidence Theory	Probability Theory	Interval Analysis
λ_1	[903.56, 910.84]	908.60	[898.20, 912.12]
λ_2	[3393.08, 3409.27]	3403.15	[3364.90, 3414.66]
λ_3	[7064.54, 7102.77]	7087.35	[7016.11,7112.78]
λ_4	[12590.59,12705.06]	12661.91	[12560.84,12720.18]

Table 3: Result comparison for DST and Probability theory with 95% guarantee probability.

Figure 4 and Table 3 show that the results given by evidence theory are interval values because of the uncertain parameters and the evidence theory result is wider than probabilistic result but much narrower than interval method result. What's more, the result calculated by probability theory is just a curve inside the resign enclosed by the belief function and the plausibility function which demonstrates the good compatibility between evidence theory and probability theory. The uncertain variables without exact probability distribution are suitable for evidence theory to handle, and it avoids the error caused by probability theory and too conservative caused by interval analysis.

5 CONCLUSIONS

A combined evidence theory and differential evolution based method for frequency analysis of structures with epistemic uncertainty in structure's stiffness properties is presented. The present method is computationally feasible, in which the differential evolution method possesses fast convergence speed and good robustness, and it is used to solve two interval optimization problems in the process of uncertainty propagation, which can greatly increase the calculation efficiency.

Also for any structural system, problems incorporating epistemic uncertainty are of great practical importance. Evidence theory with a flexible framework can formulate various basic probability assignment structures and can represent many types of uncertain data, and it has good compatibility with probability theory, so evidence theory has the potential to handle epistemic uncertainty quantification.

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REFERENCES

[1] W.L. Oberkampf, J.C. Helton, K. Sentz, Mathematical representation of uncertainty. *AIAA Non-Deterministic Approaches Forum*, 2001.

[2] L.A. Zadeh, Fuzzy sets. *Information and control*, 8, 338-353, 1965.

[3] D. Dubois，H.M. Prade，Plenum press, *Possibility theory: an approach to computerized processing of uncertainty*, New York, 1988.

[4] R.E. Moore, Prentice-Hall, *Interval analysis*, Englewood Cliffs, 1966.

[5] M. Modares, R. L. Mullen, R. L. Muhanna, Natural frequencies of a structure with bounded uncertainty, *Journal of engineering mechanics*, 132, 1363-1371, 2006.

[6] W. Gao, C.M. Song, T.L. Francis, Probabilistic interval analysis for structures with uncertainty. *Structural Safety*, 32, 191-199, 2010.

[7] A.P. Dempster, Upper and lower probabilities induced by a multivalued mapping. *The annals of mathematical statistics*, 38, 325-339, 1967.

[8] G. Shafer, Princeton university press, *A mathematical theory of evidence*, Princeton, 1976.

[9] M. Oberguggenberger, W. Fellin, Reliability bounds through random sets: non-parametric methods and geotechnical applications. *Computers & Structures,* 86, 1093-1101, 2008.

[10] S. McClean, B. Scotney, Using evidence theory for the integration of distributed databases. *International Journal of Intelligent Systems*, 12, 763-776, 1997.

[11] J. François, Y. Grandvalet, T. Denoeux, J.M. Roger, Addendum to resample and combine: an approach to improving uncertainty representation in evidential pattern classification. *Information Fusion*, 4, 235-236, 2003.

[12] X.M. Zhang, H.Y. Wang, D. Huang, Cooperative Multi-platform Data Fusion Based on DS Evidence theory. *Computer Engineering*, 33, 242-243, 2007.

[13] H.R. Bae, R.V. Grandhi, R.A. Canfield, Sensitivity analysis of structural response uncertainty propagation using evidence theory. *Structural and Multidisciplinary Optimization*, 31, 270-279, 2006.

[14] Y.C. Bai，X. Han, C. Jiang, J. Liu, Comparative study of metamodeling techniques for reliability analysis using evidence theory. *Advances in Engineering Software*, 53, 61-71, 2012.

[15] R. Storn，K. Price, Differential Evolution–A Simple and Efficient Heuristic for global Optimization over Continuous Spaces. *Journal of global optimization*, 11, 341-359, 1997.

[16] Y. Su, H.S. Tang, S.T. Xue, C.Y. Hu, Mixed aleatory-epistemic uncertainty quantifica tion using evidence theory with differential evolution algorithm. *Proceeings of the 2nd International Conference on Civil Engineering and Building Materials, CEBM 2012*, Hong Kong, 2012.

[17] L.X. Deng, H.S. Tang, C.Y. Hu, S.T. Xue, Evidence Theory and Differential Evolution for Uncertainty Quantification of Structures. *Applied Mechanics and Materials*, 249, 1112-1118, 2013.

[18] Z.P. Qiu，X.J. Wang, Comparison of dynamic response of structures with uncertain-but-bounded parameters using non-probabilistic interval analysis method and probabilistic approach. *International Journal of Solids and Structures*, 40, 5423-5439, 2003.

[19] M. Modares，R.L. Mullen，R.L. Muhanna, Natural frequencies of a structure with bounded uncertainty. *Journal of engineering mechanics*, 132, 1363-1371, 2006.