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EFFICIENT ITERATIVE SOLUTION TECHNIQUES FOR FINITE ELEMENT REANALYSES IN THE CONTEXT OF MONTE CARLO SIMULATION

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Abstract. *Several computational methods and techniques have been developed over the past decades to efficiently solve large-scale systems with thousands or even millions of equations. Certain advanced applications require successive solutions of linear systems, in order to account for changes in the problem parameters among successive simulations. Such sequences of equation systems are encountered in structural mechanics Finite Element (FE) problems with random/stochastic properties handled in the context of Monte Carlo (MC) simulation. An important feature in a MC simulation-based FE procedure is that each of the terms in the structure's stiffness matrix varies in the successive linear systems to be solved, however the differences between successive stiffness matrices are relatively small. Problems with multiple left-hand sides of this type are known as reanalysis problems.*

This paper overviews an established method for solving reanalysis problems, which is a customized version of the iterative Preconditioned Conjugate Gradient (PCG) method, and proposes a new solution method, which employs a Fixed Point Iteration (FPI) approach. The FPI-based method uses a recursive matrix equation to update the solution vector for each linear system processed. Its computational performance is enhanced by applying a relaxation procedure. A suitable preconditioning matrix is successfully used with the iterative PCG- and FPI-based methods.

A stochastic shell FE problem with 19800 degrees of freedom is studied using the MC simulation method. In order to attain sufficiently accurate probabilistic results for this test problem, 2800 MC simulations need to be performed, which give rise to a sequence of 2800 linear systems to solve. It is demonstrated that these systems can be efficiently solved with the PCG- and FPI-based reanalysis methods, which are comparatively assessed with respect to their computational performance. According to the numerical results obtained, when appropriate relaxation is applied, the simple FPI-based solution approach exhibits comparable computational behavior with the established and more involved PCG-based alternative.

1 INTRODUCTION

The computational burden associated with large-scale probabilistic/stochastic Finite Element (FE) analyses may be substantially reduced by appropriately handling the most demanding tasks in terms of processing power and storage space needs. The Monte Carlo (MC) simulation technique, which is the most effective and widely applicable method for handling large-scale probabilistic/stochastic FE problems with complicated structural response, involves expensive computations due to the successive analyses required. More specifically, assuming a linear static problem, successive linear systems of equations with multiple left- and/or right-hand sides have to be processed, since the stiffness matrix and/or the load vector change in every simulation. A standard direct method based on Cholesky factorization can be employed for solving such equations, however, it exhibits poor performance for large-scale problems and may lead to practically infeasible computations (in terms of required computing time) when the number of MC simulations to be performed is not small.

Alternatively, customized versions of the iterative Preconditioned Conjugate Gradient (PCG) method can be used. PCG can be adapted to the special features of nearby problems encountered in FE reanalyses. More specifically, PCG can be customized to take into account the relatively small differences between stiffness matrices in successive simulations, avoiding this way the treatment of each simulation's system as a stand-alone problem. PCG-customization is realized through appropriate preconditioning approaches employed to accelerate PCG convergence during the successive FE solutions.

Another iterative solution approach introduced in this work employs a Fixed Point Iteration (FPI) method to solve the FE problem at each MC simulation. This iterative method is based on the rationale of the classical Jacobi, Gauss-Seidel and Successive Over-Relaxation (SOR) methods for solving linear systems of equations. Such methods use a recursive matrix equation to generate a sequence of solution approximations. The adopted FPI implementation takes the special form of nearby FE problems into account, in order to provide efficiently computed solutions. Moreover, appropriate relaxation is employed to enhance the efficiency of the FPI method.

The remainder of this paper is organized as follows. Section 2 overviews the form of equations that arise in probabilistic/stochastic FE problems treated using the MC simulation technique. Customized PCG- and FPI-based iterative solution methods for such equations are presented in sections 3 and 4, respectively. Numerical investigation results are reported and discussed in section 5. Finally, some concluding remarks are given in section 6.

2 REANALYSIS PROBLEMS IN MC SIMULATION-BASED STOCHASTIC FE ANALYSIS

In MC simulation-based FE analysis of structures with random properties successive linear systems with multiple left-hand sides have to be processed, since the stiffness matrix $\mathbf{K}$ changes in every simulation. Assuming deterministic loads and a linear static FE problem, a system of equations of the following form needs to be solved at each simulation i (i=1,2,…,n_{sim}):

$$\mathbf{K}_i \mathbf{x}_i = \mathbf{b} . \tag{1}$$

In the above equation, $\mathbf{K}_i$ is the stiffness matrix associated with the i-th MC simulation, $\mathbf{x}_i$ is the corresponding vector of unknown nodal displacements and $\mathbf{b}$ is the vector of nodal loads. If $\mathbf{K}_0$ is the stiffness matrix associated with the initial simulation, Eq. (1) can be written as:

$$(\mathbf{K}_0 + \Delta\mathbf{K}_i)\, \mathbf{x}_i = \mathbf{b} . \tag{2}$$

Matrix $\Delta\mathbf{K}_i$ is the difference between the stiffness matrices $\mathbf{K}_0$ and $\mathbf{K}_i$:

$$\Delta \mathbf{K}_i = \mathbf{K}_i - \mathbf{K}_0 . \tag{3}$$

In a probabilistic/stochastic FE application, matrix $\Delta \mathbf{K}_i$ generally has terms with small values compared to those of $\mathbf{K}_0$. The n_{sim} Eqs. (1) or (2) solved during MC simulations form a set of reanalysis-type or nearby problems [1-3].

The standard direct method based on Cholesky factorization remains even today the most popular solution scheme for FE equations. This solution method is applied by first factorizing the stiffness matrix and then solving for the displacements vector through forward/backward substitutions. Such a solution scheme is generally stable and effective, however, it is often inefficient when confronted with stiffness matrices having large bandwidths. Moreover, a standard direct solution method treats each of the systems of the form of Eqs. (1) or (2) as a stand-alone problem and requires a computationally expensive factorization procedure to be performed at each MC simulation.

In order to overcome the deficiencies of the direct approach, specialized solution methods have been and are still being developed for reanalysis problems. They are known as reanalysis methods and are typically iterative approaches. Reanalysis methods attempt to numerically take advantage of the fact that there are relatively small differences between stiffness matrices in successive analyses. Hence, successive analyses are linked with each other and are not inefficiently treated as stand-alone problems. This way, overall computations are drastically accelerated.

The specialized handling of probabilistic/stochastic FE equations is an issue investigated in a number of publications dealing with MC-based and other probabilistic/stochastic formulations (e.g. [1,2,4-12]). In the present work, an established PCG-based solution method is overviewed and a new FPI-based solution method is introduced. The two solution approaches are assessed and compared using a numerical example.

3 THE PCG-$\mathbf{K}_0$ SOLUTION METHOD

A customized PCG version (PCG-$\mathbf{K}_0$) for the solution of reanalysis problems of the form of Eqs. (1) or (2) is obtained by applying a suitable preconditioner at each iteration to accelerate PCG convergence during the successive FE solutions [1,2]. For this purpose, the FE equations (1) are replaced by the equivalent system:

$$\tilde{\mathbf{K}}_i^{-1} \mathbf{K}_i \mathbf{x}_i = \tilde{\mathbf{K}}_i^{-1} \mathbf{b} , \tag{4}$$

in which the preconditioning matrix $\tilde{\mathbf{K}}_i$ is intended to be an approximation to $\mathbf{K}_i$. Following the rationale of incomplete Cholesky preconditionings [13], the preconditioner $\tilde{\mathbf{K}}_i$ for each MC simulation i is written as:

$$\tilde{\mathbf{K}}_i = \tilde{\mathbf{L}}_i \tilde{\mathbf{D}}_i \tilde{\mathbf{L}}_i^t = \mathbf{K}_0 + \Delta \mathbf{K}_i - \mathbf{E}_i , \tag{5}$$

where $\tilde{\mathbf{D}}_i$ is a diagonal matrix, $\tilde{\mathbf{L}}_i$ is a lower triangular matrix with unit elements on the leading diagonal and $\mathbf{E}_i$ is an error matrix. Hence, the preconditioner $\tilde{\mathbf{K}}_i$ is defined through the incomplete factorization of the stiffness matrix $\mathbf{K}_i = \mathbf{K}_0 + \Delta \mathbf{K}_i$. The error matrix $\mathbf{E}_i$ is chosen as $\mathbf{E}_i = \Delta \mathbf{K}_i$ for any MC simulation i and does not have to be formed. For the particular choice made for $\mathbf{E}_i$, Eq. (5) yields $\tilde{\mathbf{K}}_i = \mathbf{K}_0$, which means that the preconditioning matrix for any simulation i is the complete factorized initial stiffness matrix $\mathbf{K}_0$. Thus, the preconditioning matrix does not change from simulation to simulation; therefore, it needs to be formed just once at the beginning for the entire MC simulation process. Assuming that the entries of matrix $\Delta \mathbf{K}_i$ have

sufficiently small values compared to those of $\mathbf{K}_0$, the PCG procedure is equipped with a strong preconditioner $\mathbf{K}_0$ for the successive solutions. This reanalysis method is named as PCG-$\mathbf{K}_0$. The use of a single preconditioner in all MC simulations provides the linking between successive PCG-$\mathbf{K}_0$ solutions and prevents the treatment of stand-alone problems.

In order to effectively apply the PCG-$\mathbf{K}_0$ method, the stiffness matrix $\mathbf{K}_0$ of the initial simulation is retained in memory in factorized form throughout the n_{sim} simulations. This way, the preconditioning step performed at each PCG iteration is actually handled as a problem with multiple right-hand sides and is efficiently solved through forward/backward substitutions. It should be mentioned that it is preferable to select $\mathbf{K}_0$ as the stiffness matrix formed using the nominal/mean values of the random parameters of the probabilistic/stochastic FE problem considered, in order to ensure that $\mathbf{K}_0$ corresponds to the 'central' region of the sampling space. Hence, the initial stiffness matrix $\mathbf{K}_0$ is assembled, factorized and stored in computer memory before starting the actual MC simulation computations.

4 THE FPI-$\mathbf{K}_0$ SOLUTION METHOD

The FPI concept forms the basis of a class of iterative solution methods for linear systems [14]. An FPI method for the solution of a linear system $\mathbf{K}\,\mathbf{x} = \mathbf{b}$ uses a recursive matrix equation of the form:

$$\mathbf{x}^{(k+1)} = \mathbf{G}\mathbf{x}^{(k)} + \mathbf{g} \tag{6}$$

to update the solution vector at an iteration (k+1) based on its values at the previous iteration (k). By adopting the matrix splitting:

$$\mathbf{K} = \mathbf{M} - \mathbf{N}, \tag{7}$$

Eq. (6) is written as:

$$\mathbf{x}^{(k+1)} = \mathbf{M}^{-1}\mathbf{N}\mathbf{x}^{(k)} + \mathbf{M}^{-1}\mathbf{b} \tag{8}$$

or:

$$\mathbf{M}\mathbf{x}^{(k+1)} = \mathbf{N}\mathbf{x}^{(k)} + \mathbf{b}, \tag{9}$$

with:

$$\mathbf{G} = \mathbf{M}^{-1}\mathbf{N} = \mathbf{I} - \mathbf{M}^{-1}\mathbf{K}, \tag{10}$$

$$\mathbf{g} = \mathbf{M}^{-1}\mathbf{b} \tag{11}$$

and $\mathbf{I}$ being the identity matrix. Equation (6) can be seen as a recursive formula solving the system:

$$(\mathbf{I} - \mathbf{G})\mathbf{x} = \mathbf{g} \tag{12}$$

or, using Eqs. (10) and (11), of the system:

$$\mathbf{M}^{-1}\mathbf{K}\mathbf{x} = \mathbf{M}^{-1}\mathbf{b}\,. \tag{13}$$

Equation (13) represents a preconditioned version of the system $\mathbf{K}\,\mathbf{x} = \mathbf{b}$ with preconditioner $\mathbf{M}$. Thus, an FPI method can be regarded as a preconditioned iterative solution technique of the original system. Depending on the particular matrix splitting specified according to Eq. (7), a corresponding FPI-based solution method of the form (8) or (9) is defined, provided that a nonsingular preconditioning matrix $\mathbf{M}$ is selected. Matrix splittings corresponding to the classical Jacobi, Gauss-Seidel, SOR and symmetric SOR (SSOR) iteration methods are given in

[14]. FPI-based methods have been used in various publications for solving systems of equations; characteristic FPI implementations can be found in [15-18].

A new FPI-based method is defined in the present paper by another suitable matrix splitting according to Eq. (7). More specifically, in order to solve the i-th system of the sequence of n_{sim} linear systems (1), the following matrix splitting is specified:

$$\mathbf{K}_i = \mathbf{K}_0 - (-\Delta\mathbf{K}_i)\,. \tag{14}$$

Equation (14) can be rewritten in the form:

$$\mathbf{K}_i = \mathbf{M} - \mathbf{N}_i \tag{15}$$

with:

$$\mathbf{M} = \mathbf{K}_0\,, \tag{16}$$

$$\mathbf{N}_i = -\Delta\mathbf{K}_i\,. \tag{17}$$

Equation (15) defines a matrix splitting of the form (7). This is a valid splitting, as the preconditioning matrix $\mathbf{M} = \mathbf{K}_0$ is nonsingular (the initial stiffness matrix $\mathbf{K}_0$ is positive definite). Thus, a new FPI-based method can be defined with a recursive matrix equation of the form of Eq. (6):

$$\mathbf{x}_i^{(k+1)} = \mathbf{G}_i\mathbf{x}_i^{(k)} + \mathbf{g}\,, \tag{18}$$

where:

$$\mathbf{G}_i = -\mathbf{K}_0^{-1}\Delta\mathbf{K}_i = \mathbf{I} - \mathbf{K}_0^{-1}\mathbf{K}_i\,, \tag{19}$$

$$\mathbf{g} = \mathbf{K}_0^{-1}\mathbf{b}\,. \tag{20}$$

Using Eqs. (19) and (20), Eq. (18) can be rearranged as follows:

$$\mathbf{x}_i^{(k+1)} = \mathbf{x}_i^{(k)} + \mathbf{K}_0^{-1}(\mathbf{b} - \mathbf{K}_i\mathbf{x}_i^{(k)})\,. \tag{21}$$

By setting the residual $\mathbf{r}$ at each FPI iteration k of each MC simulation i as:

$$\mathbf{r}_i^{(k)} = \mathbf{b} - \mathbf{K}_i\mathbf{x}_i^{(k)}\,, \tag{22}$$

Eq. (21) becomes:

$$\mathbf{x}_i^{(k+1)} = \mathbf{x}_i^{(k)} + \mathbf{K}_0^{-1}\mathbf{r}_i^{(k)}\,. \tag{23}$$

Equation (23) gives the final form of the recursive matrix equation of the proposed FPI-based method with preconditioner $\mathbf{M} = \mathbf{K}_0$, which remains the same throughout the n_{sim} simulations. This new reanalysis method is named as FPI-$\mathbf{K}_0$. As in PCG-$\mathbf{K}_0$, the linking between successive FPI-$\mathbf{K}_0$ solutions is realized with the use of a single preconditioner $\mathbf{K}_0$ in all n_{sim} MC simulations, which needs to be formed just once.

The performance of the FPI-$\mathbf{K}_0$ method can be enhanced by applying a relaxation procedure. Examples for the use of relaxation in FPI methods are provided in [19,20]. In the present paper, relaxation is applied by updating the solution as follows:

$$\mathbf{x}_i^{(k+1)} = \omega\mathbf{x}_i^{(k+1)} + (1-\omega)\mathbf{x}_i^{(k)}\,, \tag{24}$$

where ω is a user-specified relaxation parameter. Thus, the result of Eq. (23) at a particular iteration can be first updated using Eq. (24) before proceeding to the next iteration. Clearly, the

choice of ω=1 deactivates relaxation, thus values ω>1 (over-relaxation) or 0<ω<1 (under-relaxation) are of real interest.

In order to check the convergence of the FPI-$\mathbf{K}_0$ procedure at each iteration k, the norm of the residual is calculated as:

$$\gamma_i^{(k)} = \left\| \mathbf{r}_i^{(k)} \right\| = \sqrt{\mathbf{r}_i^{(k)^t} \mathbf{r}_i^{(k)}} \,. \tag{25}$$

Note that the residual $\mathbf{r}_i^{(k)}$ is evaluated anyway through Eq. (22), in order to update the solution vector through Eq. (23), which means that the residual vector is readily available for the calculation of Eq. (25). Convergence is then assumed when the following condition holds:

$$\frac{\gamma_i^{(k)}}{\gamma_i^{(0)}} \le \varepsilon \,, \tag{26}$$

where ε is a user-specified convergence tolerance. The same convergence criterion is utilized also in the PCG method [2]. However, in PCG the residual vector at each iteration is not directly calculated through an equation of the form of (22), but through a vector update operation, which provides an estimation of the residual.

5 NUMERICAL EXAMPLE

The numerical test example used to evaluate and compare the PCG-$\mathbf{K}_0$ and FPI-$\mathbf{K}_0$ solution methods is a cylindrical shell (Fig. 1) with stochastic modulus of elasticity and thickness. This thin cylinder is supported by rigid diaphragms at its two circular edges. The structure is subjected to two concentrated loads, which are deterministic and act compressively at two mid-height opposite points of the cylinder. The MC simulation technique is employed to perform stochastic FE analyses for the cylindrical shell. The relatively fine mesh of Fig. 1 with 100×35 nodes and 6800 elements resulting in 19800 active degrees of freedom is used in all test runs for performing FE calculations. This test example has been examined also in [2,10], where it is described in more detail.

Assuming that the shell is converted to a two-dimensional domain by 'unfolding' the cylinder, the spatial randomness of the structure's modulus of elasticity E and thickness t is represented at each domain point (x,y) by two non-correlated two-dimensional univariate (2D-1V) homogeneous Gaussian stochastic fields with coefficients of variation σ_E and σ_t, respectively:

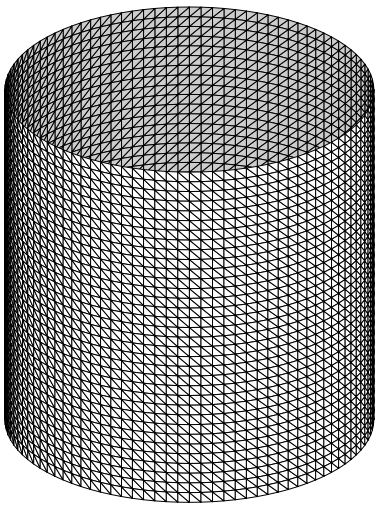

Figure 1: The FE mesh used for the shell test problem.

$$E(x,y) = E_0\left[1 + f_E(x,y)\right], \tag{27}$$

$$t(x,y) = t_0\left[1 + f_t(x,y)\right]. \tag{28}$$

In the above expressions, E_0 and t_0 are the mean values of E and t, while the zero-mean random fields $f_E(x,y)$ and $f_t(x,y)$ describe the variation of E and t about E_0 and t_0, respectively. Stochastic field samples for the MC simulations are generated using the spectral representation method [21], while discretized random field values at the elements' centroids are obtained with the local average approach [22-24]. Two independent discretized random field samples need to be produced at each MC simulation (one for E and another for t). To reduce the computational cost of producing random field values for all finite elements at each MC simulation, a coarse 'stochastic mesh' is utilized to economically generate random field values at its elements' centroids. Then, a fast bivariate interpolation procedure is employed to 'transfer' the generated random field values to the elements' centroids of the fine 'structural mesh' of Fig. 1, which is used to carry out all conventional FE calculations. The two stochastic fields for E and t have a common correlation length b=2.4m along all directions, therefore a coarse stochastic mesh with only 15×5 nodes and 120 elements is adequate for the purposes of this study. Details and justification about this two-mesh procedure and the selection of the stochastic mesh are provided in [2].

The aim for this test example is to calculate the failure probability P_f, which is defined as the probability that the absolute value of the cylindrical shell structure's radial displacement u_r at a loaded node exceeds a pre-selected critical value $u_{r,\mathrm{cr}}$, i.e. $P_f=P(|u_r|>u_{r,\mathrm{cr}})$. Hence, a MC simulation-based stochastic FE approach is applied to obtain P_f-results. A full linear FE analysis needs to be carried out at each MC simulation, in order to determine the displacement u_r and compare it with the threshold $u_{r,\mathrm{cr}}$ and eventually estimate the failure probability as $P_f=n_{\mathrm{fail}}/n_{\mathrm{sim}}$, where n_{fail} is the number of simulations with $|u_r|>u_{r,\mathrm{cr}}$. For the values $\sigma_E=\sigma_t$=5% and $u_{r,\mathrm{cr}}$=23mm adopted in this work, the total number of simulations required to achieve a sufficiently accurate P_f-result is n_{sim}=2800, as justified in Fig. 2. Thus, a sequence of 2800 linear systems of the form (1) needs to be efficiently solved.

Table 1 presents the computational performance of the PCG-$\mathbf{K}_0$ and FPI-$\mathbf{K}_0$ methods for solving the stochastic shell problem (timing results were obtained on a standard desktop PC). A convergence tolerance $\varepsilon=10^{-3}$ is used in all runs with the PCG- or FPI-based solution schemes. The results of the conventional direct solution approach are given for comparison purposes.

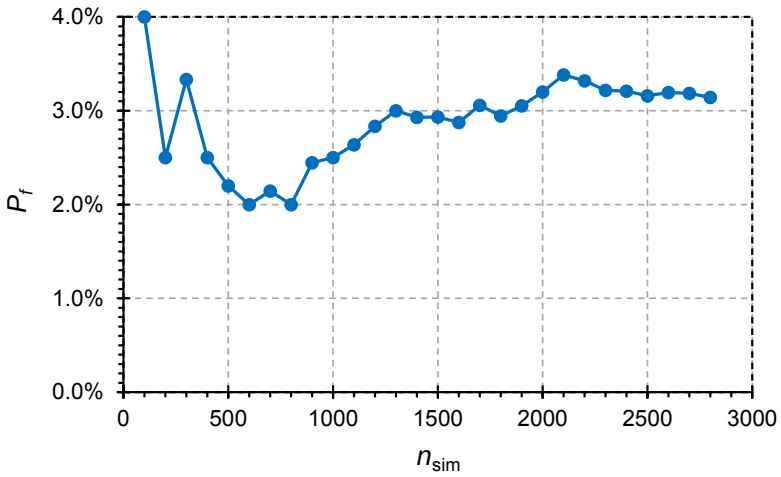

Figure 2: P_f as a function of the number of simulations.

Solution method	ω	Total number of iterations	Time (s)				Total storage (MB)
			Random field samples & stiffness matrices	Factori-zation	Solution	**Total**	
Direct			950.6	1555.1	42.6	**2548.3**	57.2
			92.8	1555.1	42.6	**1690.5**	57.2
PCG-$\mathbf{K}_0$		8139	95.3	0.6	110.2	**206.1**	34.6
FPI-$\mathbf{K}_0$	1.00	8986	95.3	0.6	121.5	**217.4**	34.5
	0.95	8862	95.3	0.6	115.2	**211.1**	34.5
	0.90	9040	95.3	0.6	122.2	**218.1**	34.5
	0.85	9411	95.3	0.6	129.3	**225.2**	34.5
	0.80	10872	95.3	0.6	146.7	**242.6**	34.5

Table 1: Iteration performance, timing results and storage requirements for various solution methods. In the first run with the direct solver reported, random field samples are generated directly on the structural mesh (no coarse stochastic mesh is utilized).

Results of a direct solution run are included, in which the stochastic mesh is the same with the structural mesh, i.e. random field samples are generated directly on the structural mesh without invoking the bivariate interpolation procedure. Table 1 gives the breakdown of timing results, in order to offer insight on the relative computing effort needed by the various tasks comprising the MC simulation-based stochastic FE analysis. More specifically, the times shown in Table 1 refer to the execution of the following tasks:

(a) generation of random field samples on the stochastic mesh, bivariate interpolation of random field values onto the actual FE mesh and formation of stiffness matrices;
(b) factorization of stiffness matrices (n_{sim} factorizations are overall required for the direct solver, but only one factorization is needed for the iterative PCG- and FPI-based schemes);
(c) solution of the resulting systems of FE equations (via forward/backward substitutions in the case of the direct method or via iterations in the case of PCG- and FPI-based techniques).

Moreover, Table 1 illustrates, depending on the employed solution technique, the total storage demands to retain all required matrix and vector data in computer memory during MC simulations. It is noted that practically the same failure probability result (P_f=3.14·10^{-2}) is attained in all runs reported in this table.

According to the results of Table 1, the iterative PCG- and FPI-based methods are clearly superior to the direct scheme. The iterative methods are faster and less storage demanding without compromising the accuracy of the final result. Such methods require only *one* stiffness matrix factorization for the preconditioner $\mathbf{K}_0$ at the initial MC simulation, while a direct solver needs a stiffness matrix factorization at each MC simulation. This huge difference in the total

number of computationally expensive factorizations required during the n_{sim} simulations explains the drastically reduced overall processing times of the iterative methods compared to the direct one.

The PCG-$\mathbf{K}_0$ and FPI-$\mathbf{K}_0$ methods exhibit similar computational performance, when FPI-$\mathbf{K}_0$ is combined with appropriate relaxation ($0.90 \leq \omega \leq 1.00$). The two methods make a difference only at the task of solving the FE equations, since they need exactly the same computations in all earlier tasks regarding the generation of random field samples, the formation of stiffness matrices and the factorization of the preconditioner $\mathbf{K}_0$. On average, both PCG- and FPI-based solution schemes perform only about 3 iterations per MC simulation, which implies that $\mathbf{K}_0$ acts as a strong preconditioner for this test problem. Compared to the FPI-$\mathbf{K}_0$ solver with ω=0.95, the PCG-$\mathbf{K}_0$ method requires in total about 9% less iterations, but leads to an overall MC simulation run, which is just 2.4% faster. This happens because the processing time per FPI-$\mathbf{K}_0$ iteration is a little lower than that per PCG-$\mathbf{K}_0$ iteration, since PCG-$\mathbf{K}_0$ performs more vector operations during each iteration, which are not required by the simpler FPI-$\mathbf{K}_0$ algorithm. Moreover, the two methods practically have the same storage demands. The storage space needed by the PCG-$\mathbf{K}_0$ method is slightly larger due to the aforementioned additional vectors processed at each iteration. Thus, a comparable computational behavior is observed between the proposed FPI-$\mathbf{K}_0$ method and the established, yet more involved, PCG-$\mathbf{K}_0$ method.

The effect of relaxation on the computational performance of FPI-$\mathbf{K}_0$ is an important aspect of this new solution method. For each problem considered, there exists an optimum ω-value that minimizes FPI iterations and leads to the most efficient MC simulation run. According to the results of Table 1, for the particular shell problem studied herein, the optimum ω-value is 0.95. Nevertheless, other ω-values in the neighbourhood of the optimum ($0.90 \leq \omega \leq 1.00$) produce similar iteration performance and efficiency. Thus, it appears that there exists a range of appropriate ω-values, which is considerably easier to identify than the exact optimum. For ω-values beyond the appropriate neighbourhood, the performance of FPI-$\mathbf{K}_0$ is negatively affected and may lead to very inefficient runs with rather high iteration counts.

It is finally noted that the low overall storage demands induced by the iterative PCG- and FPI-based methods are due to the economical way, with which stiffness and preconditioning matrices are retained in computer memory. Specifically, the stiffness matrix $\mathbf{K}_i$ at each simulation i is stored using a compact storage scheme, according to which only the non-zero stiffness terms are retained in memory. The preconditioning matrix $\mathbf{K}_0$ is stored using the classical skyline storage scheme, which retains also non-zero stiffness terms that are, however, filled-in during factorization. Floating point stiffness data are stored using double precision arithmetic for $\mathbf{K}_i$. For $\mathbf{K}_0$, single precision arithmetic is preferred instead, since the preconditioning matrix is intended to be an approximation to $\mathbf{K}_i$ anyway and high-precision storage of $\mathbf{K}_0$ does not improve the convergence behavior of either iterative method. The economical compact storage of $\mathbf{K}_i$ and the single precision storage of floating point preconditioning data minimize overall storage demands during the MC simulation procedure. Such storage approaches have been successfully applied also in [2,10]. On the other hand, the direct solution method needs double precision storage of the stiffness terms in $\mathbf{K}_i$ according to the memory demanding skyline scheme, in order to facilitate the factorization of $\mathbf{K}_i$ at each simulation i. This creates a substantially larger need for memory space compared to the iterative PCG- and FPI-based solution approaches.

6 CONCLUSIONS

This paper is concerned with the efficient solution of systems of FE equations in the framework of MC simulation using customized reanalysis methods. Iterative solution methods are

employed for this purpose, which can effectively handle sequences of linear systems of equations with multiple left-hand sides. A new method is proposed herein, which performs fixed point iterations using a recursive matrix equation to update the solution vector for each linear system processed. Through appropriate preconditioning, this FPI-based method successfully exploits the fact that there are relatively small differences between the MC simulations' stiffness matrices and achieves satisfactory convergence rate and high overall computational efficiency. In fact, its convergence behavior and its computing time and storage demands appear to be comparable with those of an established and more involved alternative solution approach based on the PCG algorithm. The effectiveness of the proposed FPI-based method is demonstrated by solving a large-scale stochastic FE problem using MC simulation, which requires the solution of a sequence of linear systems with 2800 left-hand sides.

A major advantage of the FPI method proposed in this work lies in the simplicity of its concept and implementation, as it actually consists of a single matrix equation, which is applied in a recursive manner. This matrix equation is optionally followed at each iteration by a vector update operation, when relaxation is applied. Hence, only basic numerical analysis and programming skills are required to replace the standard solver of FE equations with an FPI solution approach. Thus, non-specialist users can benefit from the computational efficiency and computer memory savings offered by such a customized reanalysis method.

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