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A CONCEPT FOR SENSITIVITY ANALYSIS AND PARAMETER CALIBRATION OF COUPLED NONLINEAR PDES AND ITS APPLICATION TO AN INDUSTRIAL GLASS FORMING MODEL

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Abstract. *Many industrial and environmental processes are characterized as complex spatio-temporal systems. Such systems, which often modeled with nonlinear coupled PDEs, are often highly complex and their relationships between model inputs, parameters and output may be poorly understood. Moreover the solutions of physics-based models commonly differ from the real measurements. Hence, aim of this work is the development of a concept which provides support in understanding of the system behavior and the parameter calibration.*

Recently the simulation considering uncertainties in models known as uncertainty quantification framework is an active research area. The uncertainty quantification framework can perform the sensitivity analysis of the uncertain influenced parameter to the quantities of interest. The probabilistic description offers also the Bayesian inference to handle the discrepancy between the model prediction and the real measurement. The efficient numerical solution is achieved by means of the generalized Polynomial Chaos expansion (gPC). With the gPC we perform the global sensitivity analysis and the model parameter calibration. We introduce the calculation of the local sensitivity analysis, which normally computed by analytical differential, by using the gPC as surrogate model.

The considered industrial process in this paper is a complex rheological forming process producing glass tubes and accordingly rods which are pre-products for optical fibers. The material parameters of the process are temperature dependent that lead to nonlinear PDEs. These dependencies are mostly empirical, thus the parameters are considered as the uncertainties of the model. The sensitivity analysis results improve the understanding of the process, i.e. the coupling of the parameters to the outputs. By the means of the concept parameters were optimal calibrated to fulfill user defined performance criteria.

1 INTRODUCTION

Nowadays the computer simulation based on mathematical model is commonly applied in every branch of natural science and engineering disciplines. Simulations are essential tools for engineers to analysis, design and control technical processes. Many industrial and environmental processes are characterized as complex spatio-temporal systems. By using physical laws, the systems can be described mathematically with Partial Differential Equations (PDEs).

In practice the modeling of the real process often leads to nonlinear coupled PDEs. Such models are often highly complex and their relationships between model inputs, model output and parameters may be poorly understood. Moreover, due to incomplete knowledge on underlying physics, simplifying assumptions or inevitable intrinsic variability, the solutions of physics-based models commonly differ from the real measurements.

These problems are widely recognized in the scientific community and have also led the uncertainty quantification (UQ) framework to constitute an active research area recently. Uncertainty Quantification covers a wide range of topics. The relevant issues are, for example, propagation of uncertainty, sensitivity analysis and inverse problem, which are mainly employed through this paper.

Under the uncertainty quantification framework, uncertainties in models are quantified using different mathematical tools. Expressing the uncertainties with a probabilistic description seems to be the one mostly chosen in practice. The stochastic approach of uncertainty modeling is achieved by representing uncertainties in the models as random variables, stochastic processes or random fields. However, solving coupled nonlinear PDEs with random variables requires generally extensive computational effort.

The generalized Polynomial Chaos has been proposed on the last few years as an efficient methodology to computing in uncertainty quantification framework. The gPC is the extension of the original Polynomial Chaos expansion (PCE), proposed by Wiener in 1938 [1]. The original Wiener's Polynomial Chaos employs Hermite polynomial to represent the Gaussian random processes. The gPC extend the PCE toward some parametric statistical non-Gaussian distribution, based on the Askey scheme of orthogonal polynomials [2].

Besides many academic examples have proven the potential of gPC, for example, the uncertainties propagation in PDE [3], calculation of Sobols' Indices for the sensitivity analysis [4] [9]and Bayesian inference in inverse problem [5], the application of gPC are found in a variety of areas, such as fluid dynamic, stricture-flow interactions, material deformations, internal combustion engine and biological problems.

In this paper, we propose a concept for sensitivity analysis and parameter calibration of coupled nonlinear PDEs based on gPC-approximation. From user defined parameter uncertainties, the system responses are expanded with Polynomial Chaos, which correspond to the uncertainty distributions. The gPC-appromation are used to perform the sensitivity analysis, and also used as surrogate model to calibrate the model parameter by given measurements. The application of this concept to the industrial glass forming model is demonstrated in this paper to show that, this approach facilitates the understanding of the relationships between model parameters and model responses. Furthermore the model parameters can be calibrated optimally to fulfill user defined performance criteria.

This paper is organized as follows. Section 2 introduced our concept of system analysis using UQ and the background of gPC. Section 3 describes our glass forming process model. Our numerical study scenario that shows the application of our concept and the results are discussed in Section 4. Finally, Section 5 presents our conclusions.

2 A CONCEPT FOR SENSITIVITY ANALYSIS AND PARAMETER CALIBRATION OF COUPLED NONLINEAR PDES BASED ON GPC-APPROXIMATION

2.1 Concept for system analysis with Uncertainty Quantification (UQ)

The computer simulation is a numerical implemented form of mathematical model, which is constructed from knowledge about the process. The simulation utilizes for gaining a new knowledge about the system behavior, which facilitates the understanding of the systems, improving model, development of control strategies and optimizing the process. Besides equations acting as model structure, the model responses are specified by model parameters. The parameters are often determined from empirical assumptions or sometimes from the estimation. Due to the complexity of nonlinear PDEs system, model parameters are commonly difficult to estimate or calibrate. Hence, aim of this work is the development of a concept which provides support in understanding of the system behavior and the parameter calibration of nonlinear PDEs system.

We introduce our concept of system analysis using UQ, as illustrated in figure 1. We focus on the knowledge about the process parameter as main target of our study. In term of science, knowledge about the process is in the form of physical law and empirical assumptions. The knowledge normally comes from the expert experience, observation of the process and measurements, which is often uncertain due to incomplete knowledge or empirical simplifying assumptions. In this work, we consider only the epistemic uncertainty, which represent the uncertainty from the incomplete knowledge. This incomplete knowledge is regarded as the ignorance, which is considered as uncertainties in our Uncertainty Quantification framework. This kind of uncertainty is known as epistemic uncertainty [6].

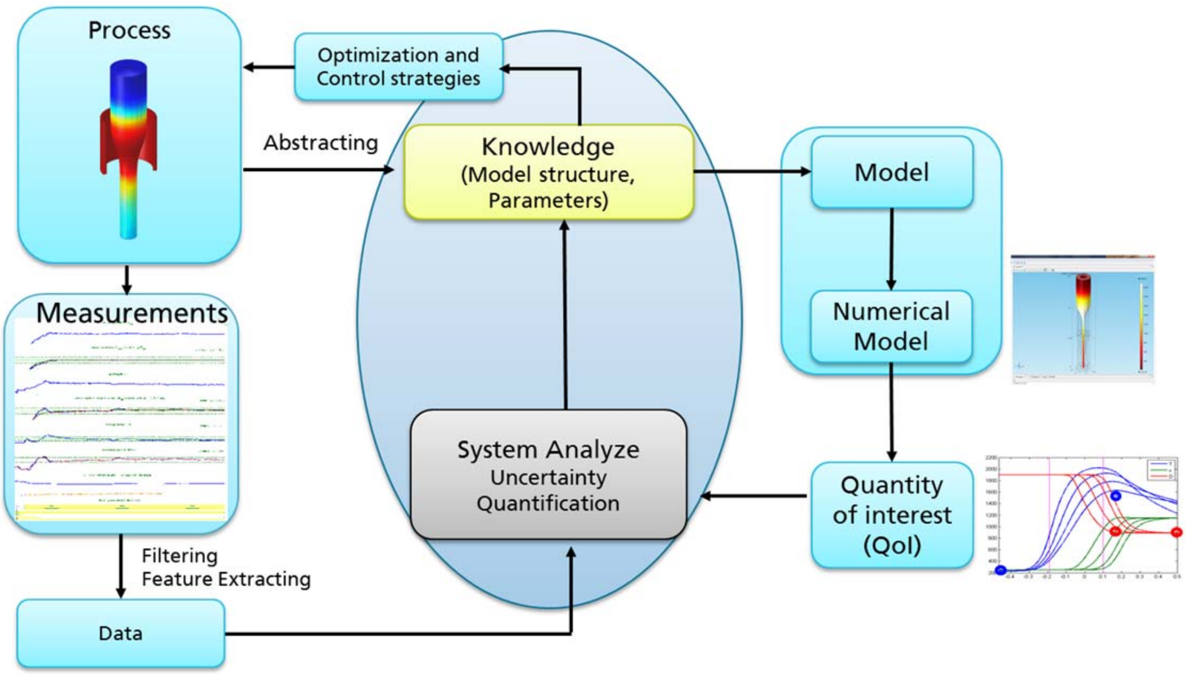

Figure 1: Concept for system analysis using UQ.

The uncertainties are expressed in form of multivariate random variables in our framework. The probability distribution of parameter uncertainties is formulated by using the principle of maximum entropy [7]. Because of the limitation of the generalized polynomial Chaos expansion, we consider in this paper only the independent random variables with the uniform, Gaussian, Gamma, and Jacobi distributions.

The system response of the model is approximated with the generalized Polynomial Chaos. The full description of the generalized Polynomial Chaos expansion can be found in [8]. In this paper we will introduce particularly the concept of gPC, which is relevant to our work.

2.2 Generalized Polynomial Chaos

Consider a deterministic nonlinear system

$$y = f(u;\theta) \tag{1}$$

Where y denotes system response, u denotes input and θ denotes the parameters of nonlinear function f. In our framework, the system response of the model y is considered as an arbitrary real-valued random variable according to some probability space$(\Omega, \mathcal{F}, \mathcal{P})$, with sample space, σ-algebra $\mathcal{F}$ and probability measure $\mathcal{P}$. In addition, it is assumed that is square-integrable, i.e. $y \in L_2(\Omega) = \{y: \mathbb{E}(y^2) < \infty\}$

The uncertain inputs and/or parameters are considered as d-dimensional multivariate random variable $\boldsymbol{X} = (X_1, X_2, \dots, X_d)^T$ with known probability distribution $p(\boldsymbol{x}) = \prod_{k=1}^{d} p(x_k)$. The random variable Y is second-order random variable as

$$Y = f(\boldsymbol{X}) \tag{2}$$

According to the stochastic spectral method, this function can be expressed in term of an infinite series of basis function of the random variable as follows:

$$Y = \sum_{|\underline{i}|=0}^{\infty} \beta_{\underline{i}} \Psi_{\underline{i}}(\boldsymbol{x}) \tag{3}$$

By using multi-index notation $\underline{i} = (i_1, i_2, \dots, i_d)$ with $|\underline{\mathrm{i}}| = \sum_{\mathrm{a}=1}^{\mathrm{d}} \mathrm{i}_\mathrm{a}$, $\Psi_{\underline{i}}()$ denotes the spectral expansion basis function and $\beta_{\underline{\mathrm{i}}}$ denotes their coefficients. In the gPC framework, these basis functions are multivariate polynomials, which are the products of the univariate orthogonal polynomial $\psi(x)$:

$$\Psi_{\underline{i}}(X_1, X_2, \dots, X_d) = \prod_{a=1}^{d} \psi_{i_a}\ (X_a) \tag{4}$$

The univariate gPC polynomials are orthogonal polynomials from Askey-scheme and determined corresponding to the marginal pdf in each dimension of the uncertain parameter as shown in Table 1.

For the determination of the coefficients β, three methods are proposed at present, namely Intrusive Galerkin, the regression and the non-intrusive spectral projection (NISP) method. Because of the difficulty of modifying our solver code, the non-intrusive spectral projection (NISP) approach is used to compute the coefficients β of gPC in our implementation. The NISP exploits the orthogonality of the gPC basis by projection to sampled model output, by taking the weighted inner product of the output gPC expansion the orthogonal polynomial

$$\beta_k = \frac{\int_{\Omega} y(x)\Psi_k(x)p(x)dx}{\|\Psi_k(x)\|^2} \tag{5}$$

In the context of NISP, the integration weights are probability density function of the random variables. Computing the inner product requires the evaluation of integrals. The inner product between system response and the polynomial can be calculated with the Gauss Quadrature formula as:

$$\int_{\Omega} y(\boldsymbol{x})\Psi_k(\boldsymbol{x})p(\boldsymbol{x})d\boldsymbol{x} = \sum_{i=1}^{n_l} y\left(\boldsymbol{x}^{(i)}\right)\Psi_k\left(\boldsymbol{x}^{(i)}\right)w^{(i)} \tag{6}$$

where $\boldsymbol{x}^{(i)}$ is quadrature nodes and $w^{(i)}$ is the associated weight, both are varied according to Quadrature formula. The Gauss Quadrature formula is closely related to the orthogonal polynomial family for the weight function $p(\boldsymbol{x})$,as shown in the Table 1. However, in case of multidimensional integral, one will confront the curse of dimensionality with Tensor product cubature formulas. The sparse grid cubature [9], which is applied in our implementation, constitutes an efficient way to moderate the curse of dimensionality of Tensor product cubature.

The system responses at all quadrature nodes are computed with the model. Then the coefficients β_k can be determined from the equation (5) and (6). By deciding the truncated order of orthogonal polynomial p, the system response can be approximated with the gPC as:

$$Y = \sum_{|\underline{i}|=0}^{\infty} \beta_{\underline{i}}\, \Psi_{\underline{i}}(\boldsymbol{X}) \approx \sum_{|\underline{i}|=0}^{N_p} \beta_{\underline{i}}\, \Psi_{\underline{i}}(\boldsymbol{X}) = Y_p \tag{7}$$

Distribution	Support	gPC basis polynomial	Weight	Quadrature
Uniform	$[a, b]$	Legendre	1	Gauss-Legendre
Gaussian	$(-\infty, \infty)$	Hermite	$e^{-(x-\alpha)^2/\beta^2}$	Gauss-Hermite
Gamma	$[0, \infty)$	Laguerre	$e^{-\alpha x}$	Gauss-Laguerre
Beta	$[a, b]$	Jacobi	$(1-x)^{\alpha}(1+x)^{\beta}$	Gauss-Jacobi

Table 1: Correspondence between the type of Generalized Polynomial Chaos, their underlying random variable distribution and the Quadrature formula.

2.3 Uncertainty Quantification with gPC

The gPC approximation as the equation (7) is applied to support analyzing the system in our work in 4 tasks.

1. Approximation of system response distribution

The probability distribution $p(y)$ can be approximated by running the Monte-Carlo sampling from $p(\boldsymbol{x})$ and then use the gPC approximation as surrogate model to compute the distribution $p(y)$. Because the gPC needs only the evaluation of polynomial, it normally requires less computation effort compare to the Monte-Carlo with complete model.

2. Parameter estimation via Bayesian inverse approach

Likewise the uncertainty propagation with Monte-Carlo approach, the parameter estimation via Bayesian inverse approach requires in general repeated solutions of the forward model to approximate a posterior probability distribution over the model parameter. In this paper we use the gPC approximation as forward model for the calculation the likelihood in the Bayesian framework to reduce the computation effort as shown in the paper [5]. The solution of the parameter estimation is the posteriori distribution of parameters, which is realized through sampling by a Markov chain Monte Carlo (MCMC) method.

3. Global sensitivity analysis

The global Sensitivity analysis (SA) is the study of the global variation of a system response of a model with regards to input parameters. Many global sensitivity analysis approaches have been proposed. The Sobol's method is a variance-based global sensitivity analysis technique that has been applied to assess the relative importance of input parameters on the output. It results the Sobol's sensitivity index as a normalized measure to determine the inputs importance. B. Sudret showed that the Sobols' indices have the relationship to the gPC coefficients [4]. According to the paper the Sobol's decomposition of the truncated gPC may be written as follows:

$$y_{P,a}(x_a) = \beta_0 + \sum_{\underline{i} \in \mathcal{A}_a^T} \beta_{\underline{i}} \Psi_{\underline{i}}(\boldsymbol{x}) \tag{8}$$

With $\mathcal{A}$ is the set of indices $\underline{\mathrm{i}}$ in the truncated expansion, $\mathcal{A}_\mathrm{a}^\mathrm{T} \coloneqq \{\underline{\mathrm{i}} \in \mathcal{A},\ \mathrm{i_a} > 0\}$, and the variance of output y respect to the inputs x_a are:

$$\sigma_a^2 = \int y_{P,a}^2(\boldsymbol{x}) p(\boldsymbol{x})\, d\boldsymbol{x} = \sum_{\underline{i} \in \mathcal{A}_a} \beta_{\underline{i}}^2 \left\|\Psi_{\underline{i}}\right\|^2 \tag{9}$$

where $\mathcal{A}_\mathrm{a} \coloneqq \{\underline{\mathrm{i}} \in \mathcal{A},\ \mathrm{i_a} > 0, \mathrm{i_{j \neq a}} = 0\}$, with the total variance

$$D = Var[Y] = \sum_{\substack{\underline{i} \in \mathcal{A} \\ \underline{i} \neq 0}} \beta_{\underline{i}}^2 \left\|\Psi_{\underline{i}}\right\|^2 \tag{10}$$

then the first order PC-based Sobol's index of the a-th input is:

$$S_a = \sigma_a^2 / D \tag{11}$$

Note that the summation of all Sobols' indices both first order and higher-order interaction indices are one.

$$\sum_{a=1}^{d} S_a + \sum_{a<j}^{d} S_{aj} + \cdots + S_{12\ldots d} = 1 \tag{12}$$

Further information about PC-based Sobols' indices can be found in [4] and[10].

4. Local sensitivity analysis

Local sensitivity analysis method involves taking the partial derivative of system response with respect to an input variable x_a at some fixed point $\boldsymbol{x^0}$ in the input space [11]. This derivative indicate the sensitivity of input parameters at fixed point on the output

$$\delta y(\boldsymbol{x^0}; x_a) = \left.\frac{\partial y}{\partial x_a}\right|_{\boldsymbol{x^0}} \delta x_a \tag{13}$$

In case that all of input parameters have a uniform distribution, the partial derivative with respect to an input variable x_a can be approximated as follows:

$$\frac{\partial y_P}{\partial x_a}(\boldsymbol{x^0}) = \sum_{\underline{i} \in \mathcal{A}_a^T} \beta_{\underline{i}} \left.\frac{\partial \Psi_{\underline{i}}(x)}{\partial x_a}\right|_{\boldsymbol{x^0}} \tag{14}$$

3 SIMULATION OF GLASS FORMING PROCESS UNDER UNCERTAINTIES

3.1 Computational model of glass forming process

Our considered industrial process in this paper is a complex rheological forming process producing glass tubes and accordingly rods which are pre-products for optical fibers production. The forming process involves a wide temperature range and is characterized by large deformations. The process setup is visualized in Figure 2(left). The cylinder is fed with slow velocity v_f in an oven where it is heated up to its forming temperature. Below the oven the tube is pulled with a higher velocity v_p resulting in thin glass rods (resp. tubes). The main physical phenomena of the glass forming process arise from radiation, heat convection, and fluid dynamics. The process is regarded as a Newtonian fluid flow with free surfaces. Basically, the model consists of two main parts describing namely the glass flow and the heat transfer in the glass and from the oven to the glass [12],[13]. The conservation laws of mass, momentum and energy formulate the PDE systems of the process.

$$\begin{aligned} \frac{\partial\rho}{\partial t} + div\,(\rho\vec{v}) &= 0 \\ \frac{\partial\rho\vec{v}}{\partial t} + (\vec{v}\cdot\nabla)\rho\vec{v} &= \nabla\cdot[-pI + \mu(T)(\nabla\vec{v} + (\rho\vec{v})^T)] + \rho\vec{g} \\ \frac{\partial\rho C_p(T)T}{\partial t} + (\vec{v}\cdot\nabla)\rho C_p(T)T &= \nabla\cdot(\lambda(T)\nabla T) \end{aligned} \tag{15}$$

In the equations (15) it denotes $\vec{v}$: velocity vector, T: temperature of the glass, μ: dynamic viscosity, C_p: specific heat capacity, λ: effective heat transfer coefficient (considers radiative heat transfer in a simplified way), ρ: density of the glass, g: gravitational acceleration,

The Navier-Stokes with free surface flow allows the calculation of the geometry deformation. The boundary conditions of the PDEs play a major role in this model as well. The geometry of the outer boundary will move with the velocity at boundary. The tracking of the moving boundary succeed by using Arbitrary Lagrangian-Eulerian (ALE) method [14]. The heat comes from the oven into the glass at outer boundary via the radiation. With the Stefan Boltzmann laws the heat flux at boundary can be written as:

$$q_{rad} = \epsilon\sigma_B(T_{oven}^4(z) - T^4) \tag{16}$$

ϵ: Emissivity of the glass, σ_B: Stefan-Boltzmann constant, $T_{oven}(z)$: oven temperature profile, which is modelled as a function of z-coordinate. Due to the temperature dependency of material parameter such as $\lambda(T)$, $C_p(T)$ and $\mu(T)$ as well as the radiative heat flux term, the PDEs are strongly nonlinear. The structure of the model, the coupling terms and nonlinearities are illustrated in Figure 1 (right).

For a 3D simulation of the glass forming process this initial-boundary value problem have to be solved numerically. The Finite Element Method is standard technique to solve PDE currently and also used here. By assuming axisymmetric the model can be reduced to a 2D model, which is discussed in this paper. More details about the glass forming Finite Element model can be found in our previous work [15], [16].

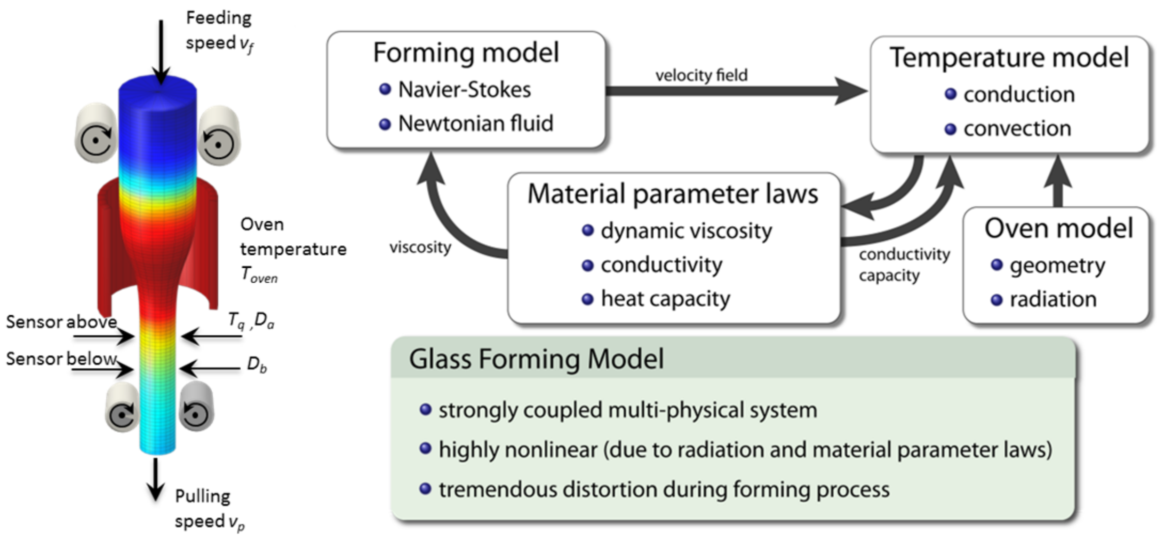

Figure 2: schematic of industrial glass forming process (left) and structure of nonlinearities of the model (right)

3.2 Simulation under uncertainties

Uncertainties in glass forming process model arise from many sources, such as inaccuracy operating parameter, environmental uncertainties, empirical determination of material parameters or model uncertainty due to simplify assumption. As the related work, Andryas Mawardi and Ranga Pitchumani have shown the numerical simulations of an optical fiber drawing process under uncertainty [17]. They showed how the uncertainties of the input parameters propagate through the fiber drawing process model. The calculation of the output distributions are achieved by simulation with the sampling of input parameters using Latin Hypercube Sampling (LHS) method.

The optical fiber drawing process is analogous to our glass forming process from physics viewpoint. However, the distinctions of the geometry dimension and the process setup make our glass forming model difference from the fiber drawing process. Moreover, our UQ framework is based on generalized Polynomial Chaos. We firstly construct the gPC-approximation model by computing model at sparse quadrature nodes. Then the gPC-model is utilized for analyzing the system in our UQ Framework as mentioned in the section 2. Our Procedure of system analysis based on gPC is summarized in figure 3. The procedure is described in details with the study scenario in the next section.

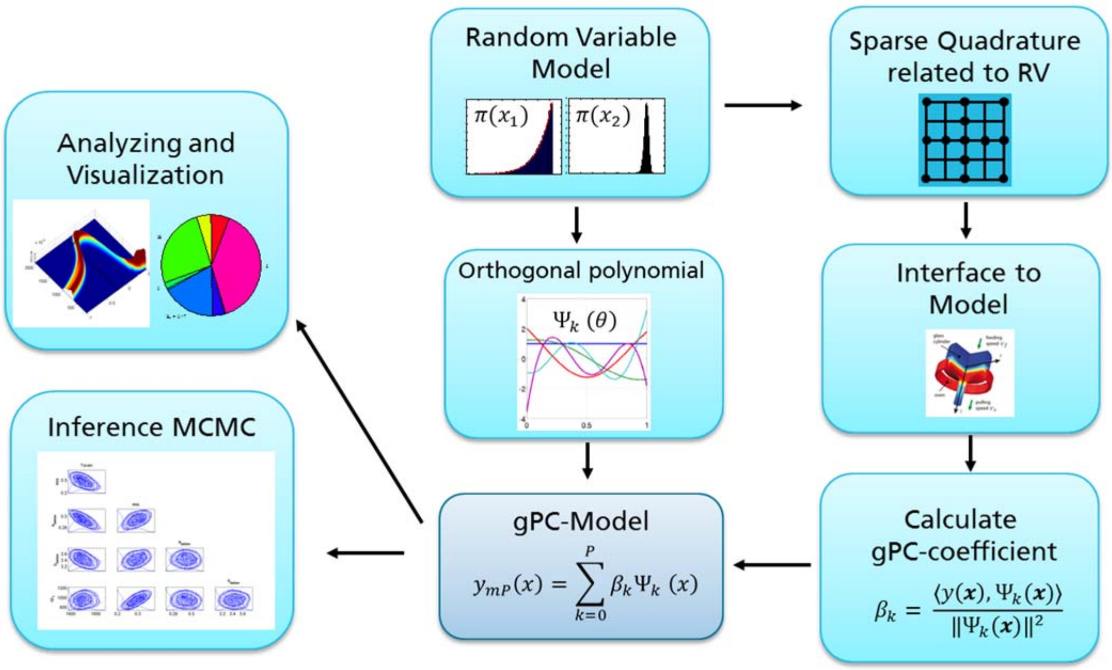

Figure 3: Procedure of system analysis based on gPC

4 STUDY SCENARIO AND RESULTS

4.1 Study scenario

In our glass forming model, there are about 30 concerned parameters. For illustrating our concept in this paper, we restrict the input parameters to 4 parameters, which are 2 material parameters and 2 oven parameters. As mentioned in the section 3, the temperature dependent material parameters are defined by the following empirical formulas:

$$\begin{aligned} C_p(T) &= C_{p1} + C_{p2}\left(T - C_{p3}\right) \\ \lambda(T) &= \lambda_1 + \lambda_2 \cdot T^{\lambda_3} \end{aligned} \tag{17}$$

From the equation (16), the oven model is defined by the emissivity ϵ and the oven temperature profile $T_{oven}(z)$ which is parameterized as:

$$T_{oven}(z) = \begin{cases} T_{offset} + T_{max} \cdot \exp\left(\frac{-|z-z_0|^{a_{below}}}{b_{below}}\right) \text{for } z \geq z_0 \\ T_{offset} + T_{max} \cdot \exp\left(\frac{-|z-z_0|^{a_{above}}}{b_{above}}\right) \text{for } z < z_0 \end{cases} \quad (18)$$

Based on our previous study, the importantly influencing parameters are C_{p2}, λ_2, a_{below} and b_{below}. We consider them uncertain and set the others parameters as deterministic variable. However, the parameters a_{below} and b_{below} are difficult to interpret into physics meaning. In order to make it more comprehensible, we consider the temperature of oven at two defined points $T_{o1} = T_{oven}(z = z_1)$ and $T_{o2} = T_{oven}(z = z_2)$ as uncertain parameters. The parameters a_{below} and b_{below}, are calculated by giving T_{o1} and T_{o2} by curve fitting method.

In summary, we consider four uncertain parameters $\boldsymbol{x} = \left[T_{o1}\ T_{o2}\ C_{p2}\ \lambda_2\right]^T$. All parameters are assumed to have a uniform distribution around the previous deterministic value with corresponding percentage range. It is noted that another distribution such as Jacobi, gamma and gauss are also possible. In addition, it is assumed that all parameters are independent so that the probability distribution of the uncertain parameters is

$$p(\boldsymbol{x}) = p(T_{o1}) \cdot p(T_{o2}) \cdot p(C_{p2}) \cdot p(\lambda_2) \quad (19)$$

For the system responses of the model, we consider:

- The shrinkage, the difference of diameter between the two sensor positions $\Delta D = D_a - D_b$
- The temperature of glass at sensor position, which is called glass temperature T_g
- The viscous force defined as: $VF(z) = \int 2\mu(T) \frac{\partial w}{\partial z} dA$ at sensor position

4.2 Results

The gPC approximation is constructed with the procedure regarding the figure 3. According to the probability distribution of the uncertain parameters $p(\boldsymbol{x})$, the sparse quadrature nodes are generated. The system responses at the quadrature nodes are computed with FEM model mentioned in the section 3.1. The gPC orthogonal polynomials are composed corresponding to the probability distribution. Using NISP approach, the polynomial coefficients are determined according to the section 2.2. The coefficients and orthogonal polynomial construct the gPC approximation model for the three system responses mentioned above to utilize our system analysis. Firstly, we approximate the probability distribution of the three system responses. The distribution can be approximated by running Monte-Carlo of gPC model with samples of $p(\boldsymbol{x})$. The figure 4 (left) shows the histogram and scatter plot of the three system responses. This plot facilitates us to comprehend the system response behavior. One can see possible values of model responses as well as their relations according to defined uncertain parameters. Please note that we cannot provide the information about the unit variables in the plot because of our partner confidential data.

To increase understanding of the relationships between input parameters and system responses, we perform sensitivity analysis with gPC model. As mentioned in the section 2.3, the Sobols' indices, which represent the sensitivities of the system responses with respect to input parameters, can be computed from the gPC coefficients. The computed Sobols' indices in our study are presented in the figure 4 (right). It illustrates the first order indices namely "main effect index" for the three system responses. It shows that the parameter λ_2 is the most importantly influencing all system responses and then follows with the parameter T_{o1}.

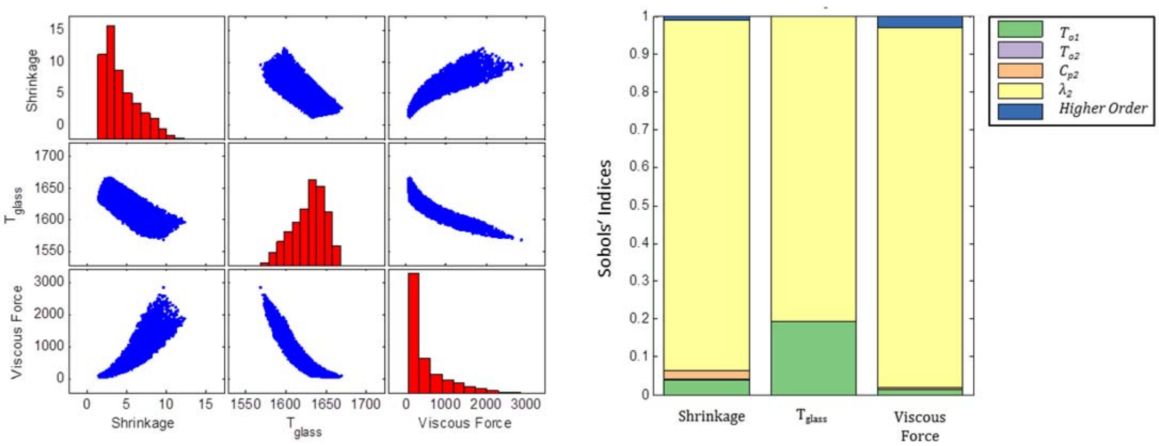

Figure 4: The histogram and scatter plot of the three system response of glass forming model represent the probability distribution by given uncertain parameters (left) and First order Sobols' sensitivity indices of the three system responses according to four uncertain parameters (right).

With our local sensitivity based on gPC approach, the derivative of model responses with respect to the parameter λ_2 can be approximated as mentioned in the section 2.3. The figure 5 shows the system responses and their derivatives subject to the parameter λ_2 by setting other uncertain parameters as deterministic. This information is very useful for the user to develop and improve the model. It is noted that this approximation of derivative work well because of the smoothness behavior of the PDE solution.

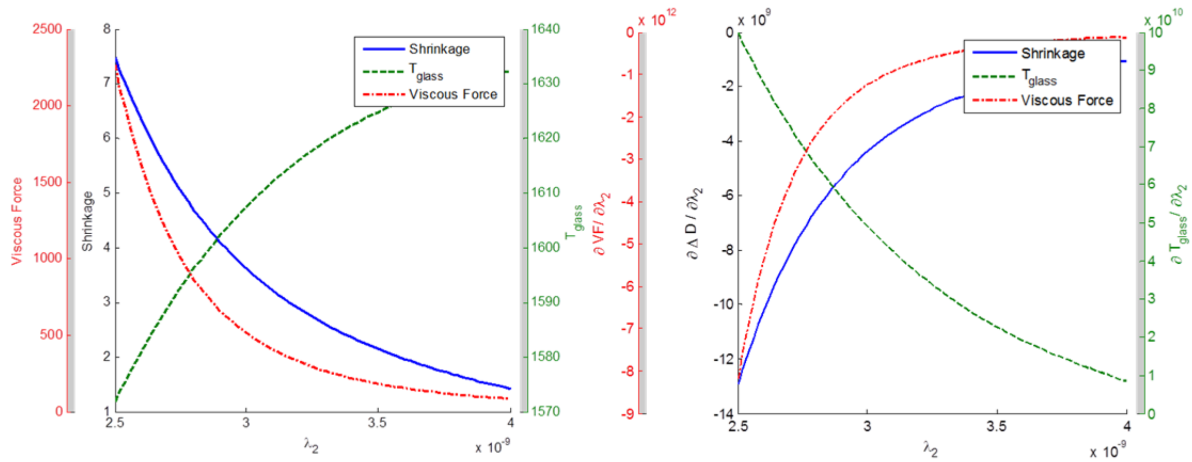

Figure 5: the three system responses subject to the parameter λ_2 by setting other parameters deterministic (left) the derivative based on gPC of the three system responses with respect to the parameter λ_2 (right).

Finally, we would like to find the parameter space, which provide the specified area in the system response space. In the UQ framework, by specified model response, which should represent some real measurement, the associated parameters can be estimated via Bayesian inference. The figure 4 (left) present the possible values of system responses. If the measurements locate outside the scatter plot, it could mean that the measurement does not belong to the defined input space or the model is deficient and have to be improved.

If the measurement locates inside the scatter plot, the gPC model can be applied as a forward model for computing the likelihood, as mentioned in the section 2.3. The figure 6 shows the result of one of our parameter estimation studies. The three system responses are specified with the noise in this case three different values of shrinkage but the same values of glass temperature and viscous force. A prior distribution is assumed as $p(\boldsymbol{x})$. The posterior distribution is accomplished via MCMC sampling. The scatter plots of the posterior distribution of uncertain parameter for three cases of the specified outputs are shown in figure 6. Because of the low sensitivity value of the parameter T_{o2}, the plots with this parameter provide no information, therefore are excluded here.

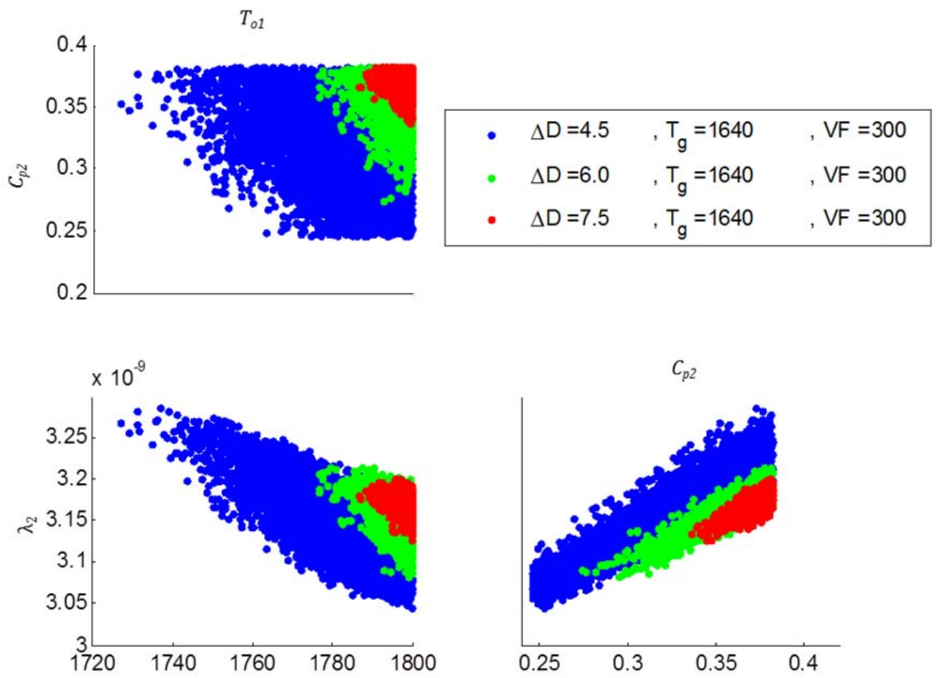

Figure 7: The sampling posterior distribution of uncertain parameter by given system responses.

5 CONCLUSION

A detailed discussion on the concept for analyzing nonlinear coupled system using uncertainty quantification was presented. The concept combines the forward uncertainty propagation, sensitivity analysis and Bayesian parameter estimation approach as a tool for analyzing the system and model parameter calibration. The generalized Polynomial Chaos is applied to reduce the computational effort of stochastic problem in UQ framework. The uncertain parameters in model are expressed as multivariate random variables with known distribution. According to the distribution, the corresponding orthogonal polynomials based on Askey scheme are constructed as a basis function. The Coefficients of the gPC polynomial are computed with NISP using sparse quadrature. The gPC approximation is used as a surrogate model in all of the UQ tasks. The application of this concept to the industrial glass forming process model was presented. The glass forming model is highly complex so that the relationships between parameters and system response are poorly understood. Moreover, the prediction of model differs from the real measurement. Our concept using UQ based on gPC improves the understanding of the process and facilitates the parameter calibration. Our procedure should expand the knowledge about the system, which facilitates improving model, development of control strategies and optimizing the process further.

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