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APPLICATION OF ASYMPTOTIC SAMPLING TO STRUCTURAL RELIABILITY PROBLEMS

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Abstract. *Monte Carlo methods are most versatile regarding applications to the reliability analysis of high-dimensional nonlinear structural systems. In addition to its versatility, the computational efficacy of Monte Carlo method is not adversely affected by the dimensionality of the problem. Crude Monte Carlo techniques, however, are very inefficient for extremely small failure probabilities such as typically required for sensitive structural systems. Therefore methods to increase the efficacy for small failure probability while keeping the adverse influence of dimensionality small are desirable.*

On such method is the asymptotic sampling *method. Within the framework of this method, well-known asymptotic properties of the reliability index regarding the scaling of the basic variables are exploited to construct a regression model which allows to determine the reliability index for extremely small failure probabilities with high precision using a moderate number of Monte Carlo samples.*

The basic concepts underlying this method are explained in detail. Its applicability is demonstrated by solving the first passage problem involving several thousands of random variables. All results are cross-checked agains crude Monte Carlo results with an extremely large sample size. It is shown that the necessary number of samples can be reduced by several orders of magnitude as compared to crude Monte Carlo.

1 INTRODUCTION

Recent developments in structural reliability are focussing on efficient Monte Carlo simulation methods capable of treating problems with several thousands of random variables. Such a large number of variables frequently arises in the discrete (spectral) representations of random processes and random fields. Classical structural reliability methods such as the First Order Reliability methods (FORM) in many cases cannot be applied due to the lack of efficient gradient evaluation, or even due to non-differentiability. Monte Carlo methods on the other hand do not require gradients, and therefore are generally very robust.

A fairly recent overview of currently available Monte Carlo simulation methods for structural reliability assessment with an attempt to evaluate their suitability for high-dimensional problems involving several thousands of random variables has been presented in the Benchmark study [10].

Specifically, in the present paper the relatively recent Asymptotic Sampling Method [2, 11, 4] is discussed in detail. This method and other recent developments in structural reliability are focussing on Monte Carlo simulation methods designed to explore the geometric properties of the safe (or failure) domain by scaling the spatial coordinates [7, 8]. This can be achieved by either expanding/shrinking the space, or by up/downscaling the standard deviations of the uncorrelated Gaussian variables.

It is demonstrated how *Asymptotic Sampling* - which provides an asymptotically unbiased estimation procedure for the failure probability - can be applied to systems reliability (series and parallel) analysis as well as the first-passage problem in random vibration analysis. Numerical examples on one hand demonstrate the high computational efficacy and on the other hand allow to quantify the inherent bias and statistical uncertainty.

2 ASYMPTOTIC PROPERTIES

Due to a result by Breitung [1] it is known that for an arbitrary reliability problem the reliability index β as computed from the First Order Reliability Method (FORM) is asymptotically exact as the reliability index tends to infinity, or alternatively, as the standard deviation of the basic variables in Gaussian space expressed as $\sigma = 1/f$ tends to zero. As stated by [1, 6], the reliability index then asymptotically depends linearly on f or, in scaled notation

$$\lim_{f \to \infty} \frac{\beta(f)}{f} = \text{const.} \tag{1}$$

In order to exploit this asymptotic relation, [2] suggested to utilize the formulation

$$\beta(f) = Af + \frac{B}{f} \tag{2}$$

In this equation, the exponent -1 of f in the second term has been chosen arbitrarily. The only requirement to satisfy the asymptotic property in Eq. 1 is that this term vanishes as $f \to \infty$. Here a slightly modified version is suggested

$$\beta(f) = Af + \frac{B}{f^p} \tag{3}$$

with $p > 0$ which can be adjusted to the specific problem at hand. In order to demonstrate these asymptotic relations, consider a two-dimensional systems reliability problem with two failure modes described by the limit state equations (cf. Fig. 1)

$$g_1(X_1, X_2) = \delta - X_1; \quad g_2(X_1, X_2) = \delta - X_2 \tag{4}$$

These limit state functions (half planes in $\mathbb{R}^2$) define two failure sets $\mathscr{F}_1$ and $\mathscr{F}_2$, respectively. Both X_1 and X_2 are initially assumed to be standard Gaussian variables.

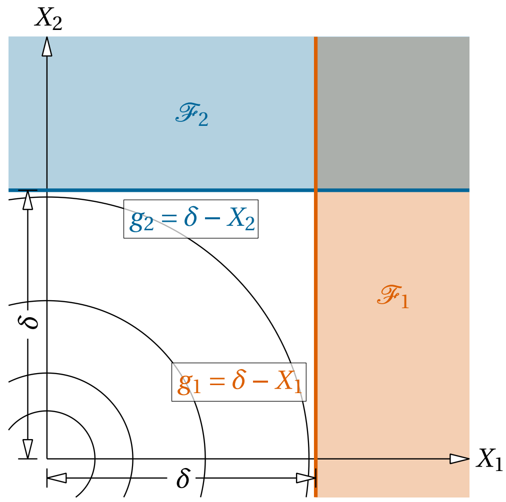


Figure 1: Systems reliability problem

For a parallel system, failure is defined by the intersection of the individual failure sets, i.e.

$$\mathscr{F}_p = \mathscr{F}_1 \cap \mathscr{F}_2 \tag{5}$$

while for series system it is defined by the union of the failure sets

$$\mathscr{F}_s = \mathscr{F}_1 \cup \mathscr{F}_2 \tag{6}$$

For both cases, the generalized reliability index $\beta = \Phi^{-1}(1 - P_F)$ can be computed analytically, i.e.

$$\beta_p = \Phi^{-1}[1 - \Phi(-\delta)^2]; \quad \beta_s = \Phi^{-1}[1 - 2\Phi(-\delta) + \Phi(-\delta)^2]; \tag{7}$$

Here $\Phi(\cdot)$ denotes the standard normal cumulative distribution function and $\Phi^{-1}(\cdot)$ is its inverse. For the case $\delta = 3$, these reliability indexes become $\beta_s = 2.78$ and $\beta_p = 4.63$. If the standard deviations of the variables X_1, X_2 are divided by a factor of f, then the reliability indexes become

$$\beta_p(f) = \Phi^{-1}[1 - \Phi(-\delta f)^2]; \quad \beta_s = \Phi^{-1}[1 - 2\Phi(-\delta f) + \Phi(-\delta f)^2]; \tag{8}$$

In view of Eq. 1, the quantities $\beta_p(f)/f$ and $\beta_s(f)/f$ are shown as functions of f in Figs. 2a and 2b, respectively, for the case $\delta = 3$. The results show that the scaled reliability index indeed approaches a constant values as $f \to \infty$.

Carrying out a least-squares regression on these functions based on Eq. 3 yields approximations shown in Figs. 2a and 2b. Both coefficients A and B as well as the exponent p were determined by the nonlinear optimization algorithm CONMIN [13] implemented in the software package slangTNG [5]. For the parallel system, it was found that $p = 0.29$ and for the series system $p = 0.62$. In both cases, the regression model fits the true curve extremely well.

An alternative view on the asymptotic properties can be obtained by studying the CDF of the safety margin $Z = g(X_1, X_2)$, specifically in the lower tails. For the series system as outlined above, a single limit state function g combining both failure modes can be written as

$$g(X_1, X_2) = \min(g_1, g_2) = \min(\delta - X_1, \delta - X_2) \tag{9}$$

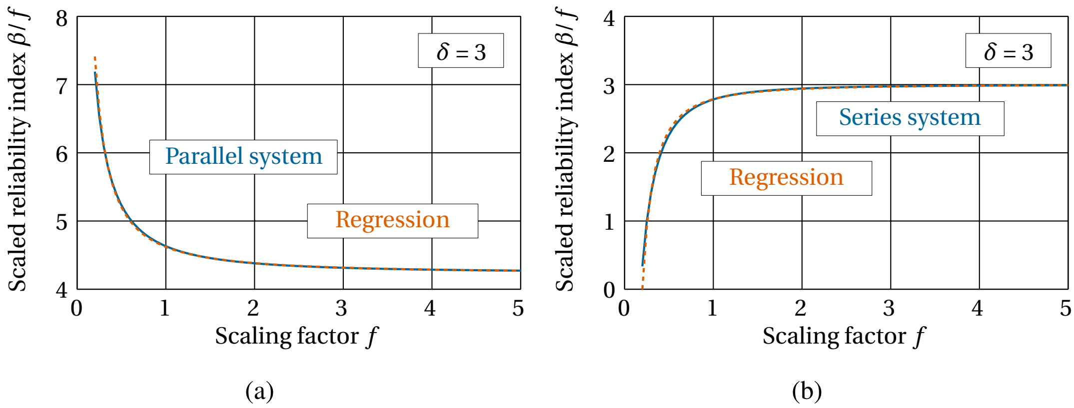


Figure 2: Exact solution and regression model for parallel system (left) and series system (right)

By introducing the safety margin $Z = g(X_1, X_2)$ we obtain a random variable whose CDF is given by

$$F_Z(z) = 1 - F_{X_1}(\delta - z) F_{X_2}(\delta - z) \tag{10}$$

Each value of the CDF can be associated with a reliability index by means of

$$\beta_z = \Phi^{-1}\left[1 - F_Z(z)\right] \tag{11}$$

If the standard deviation of the random variables are divided by a scaling factor f, then the above expressions become

$$F_Z(z, f) = 1 - F_{X_1}(\delta f - z f) F_{X_2}(\delta f - z f) \tag{12}$$

and the scaled reliability index is

$$\frac{\beta_z(f)}{f} = \frac{1}{f}\Phi^{-1}\left[1 - F_Z(z)\right] \tag{13}$$

Fig. 3 shows this function for different values of the scaling factor f for the series system. For the parallel system, the analogous results are shown in Fig. 4. Additionally, these figure contain Monte-Carlo simulation results based on 200 samples. It can be seen that the curves have an excellent agreement. It should be noted that for improved statistical stability, an advanced simulation technique based on Sobol sequences [12] has been utilized. In Figs. 3 and 4, the failure probability can be found from the scaled reliability index at $z = 0$ and $f = 1$. It can be seen that this point is hardly reached by the Monte Carlo samples (especially for the parallel system).

A somewhat different representation is obtained by plotting the value of the CDF (expressed in terms of the scaled reliability index) as a function of the safety mating z and the scaling parameter f. For the series systems, this is shown on Fig. 5a. Once these asymptotic relations are established, the asymptotic sampling procedure attempts at estimating the regression curves from Monte Carlo samples. Due to the sampling uncertainty, estimates for $\beta(f)$ are more stable for small values of f, or conversely, for a given sampling error margin fewer samples are required. This implies, that typically the regression curve will be based on samplings with factors $f < 1$ and then be used to extrapolate the curve $\beta(f)$ to the value $f = 1$ (which corresponds to the actual situation).

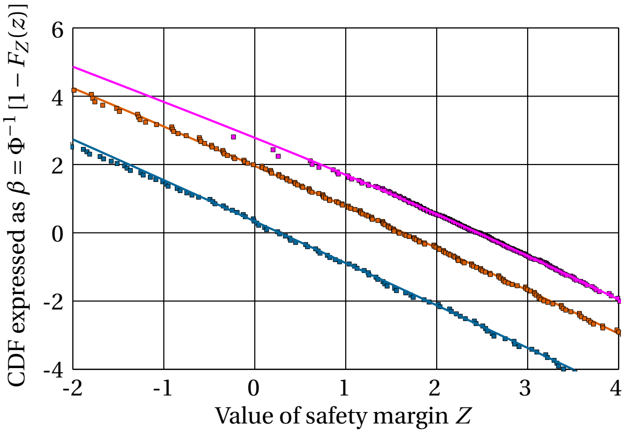


Figure 3: CDF of safety margin expressed as scaled reliability index for series system

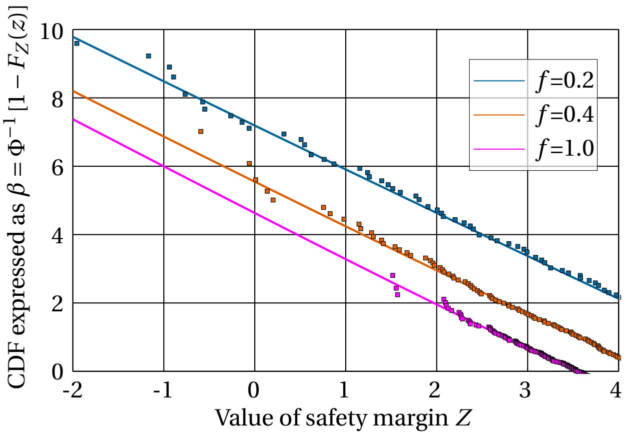


Figure 4: CDF of safety margin expressed as scaled reliability index for parallel system

The same function is then obtained by a regression over the Monte-Carlo samples obtained from 5 different values of f, namely $f = 0.1, 0.4, 0.6, 0.8, 1.0$. The sample size is 500 for each value of the scaling factor f. The regression function chosen is

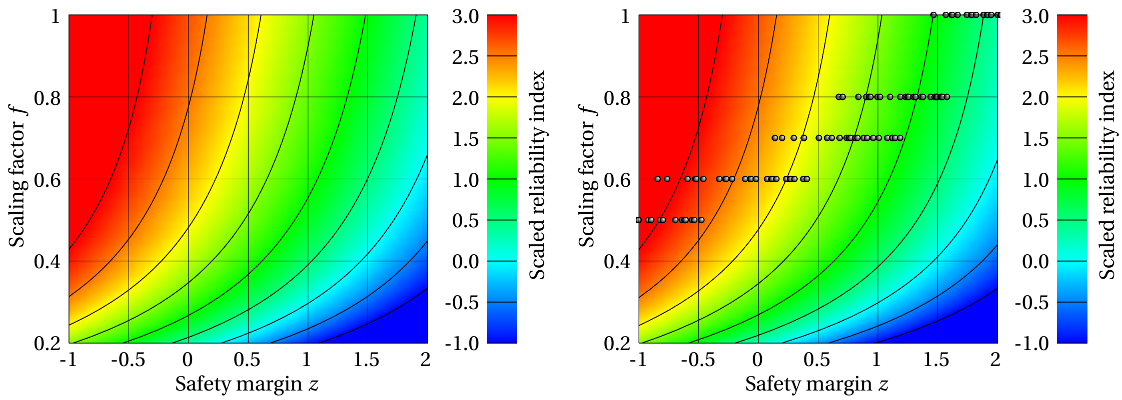
$$\hat{\beta}(z, f) = c_1 f + c_2 f^{-p} + c_3 z + c_4 z f^{-p} \tag{14}$$

with p fixed at the value of 1.3. This result is shown in Fig. 5b. The reliability index for the original series system is then obtained by evaluation the expression for $\hat{\beta}(z, f)$ at z =0 and f =1. This results in $\hat{\beta}_s = 2.74$. For the parallel system, the same analysis yields the results as shown in Fig. 6a and 6b. In these figures, the locations of the support points of the regression analysis are indicated by dots. The reliability index for the actual system is $\hat{\beta}_p$ =4.66. Both reliability indexes are very close to the exact results given above.

3 FIRST PASSAGE PROBLEM

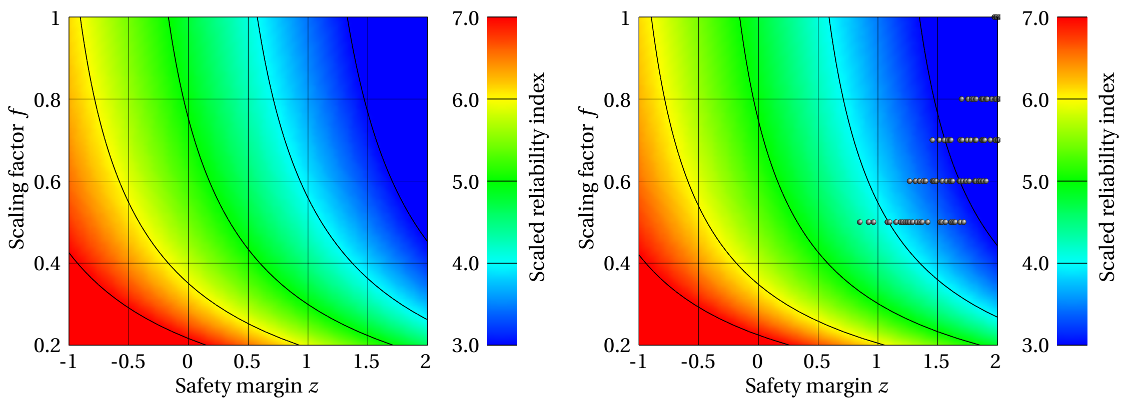
3.1 Problem definition

For applications in time-dependent problems as those arising structural dynamics, the appropriate reliability formulation leads to the first-passage problem. Here the failure domain is



(a) (b)

Figure 5: Exact solution (left) and regression model (right) for series system



(a) (b)

Figure 6: Exact solution (left) and regression model (right) for parallel system

typically written as

$$D_F = \{(X_1, \ldots X_n) \mid \max_{1 \le i \le N} h_i(X_1, \ldots X_n) \ge \xi\} \tag{15}$$

In this equation, $h(.)$ denotes a response quantity of interest, and ξ is a critical threshold value of this response. The random variables X_i usually denote the random excitation (e.g. earthquake or wind) which is discretized in time. A schematic sketch is shown in Fig. 7. Note that due to the principle of causality, the values of h at point i in time can only depend on the basic variables $X_1 \ldots X_i$ (i.e. on those in the past put to the present as expressed by the index i), and not on those in the future. Also note that Eq. 15 formally defines the failure of a series system with a large number of individual failure domains $\mathscr{F}_i$.

3.2 Linear SDOF system

As a simple first-passage problem, consider the transient response of an SDOF oscillator governed by the differential equation

$$m\ddot{x} + c\dot{x} + kx = w(t) \tag{16}$$

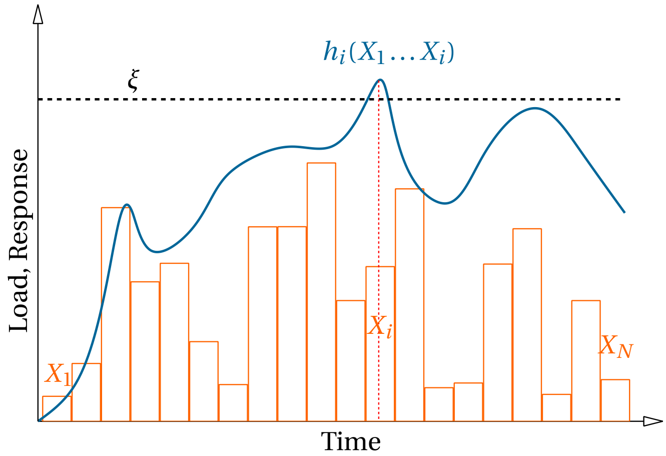


Figure 7: Schematic sketch of first passage problem

in which $w(t)$ is a stationary white noise with intensity D_0. For the numerical treatment, the white noise is replaced by a sequence of i.i.d. normal variables X_i spaced uniformly at time interval δt. The variables have zero mean and a variance $\sigma_F^2 = \frac{D_0}{\Delta t}$. Numerical values are k = 1 N/m, m = 1 kg, c = 0.1 kg/s, $D_0 = \frac{\pi}{50}$ m^2/s^3, Δt = 0.15 s. The total time considered is T = 40 s. For reference, the stationary limit of the standard deviation of the displacement response is σ_X = 0.5605 m. The failure criterion considered is that the maximum absolute value of the response $x(t)$ exceeds a threshold value ξ anywhere in the entire time duration $[0, T]$. Threshold values considered are ranging from 1 to 5 m (i.e. up to about 9 times the stationary standard deviations).

The asymptotic sampling procedure is carried out using 5 runs of 1000 simulations each. The runs have different scaling factors f for the excitation (ranging from 1 to $1/3$) thus covering different regions of the first passage probability. Note that these 5000 simulations cover the entire range of threshold values ξ considered simultaneously, i.e. the analysis does not have to be repeated for different threshold values.

Fig. 8 shows the distribution function of the peak response expressed in terms of the scaled reliability index β/f for different values of the scaling factor f. It can clearly be seen that significantly larger values of the response (and therefore larger range of the distribution function) can be reached by reducing the value of f. For f = 1, the reachable threshold value ξ is approximately 2 with a sample size of 1000. For f = 0.5, the reachable threshold is about 4.

The results of the regression based on Eq. 14 are shown in Fig. 9. For reference, the results of the Monte Carlo run with f = 1 are shown in this figure as well. As mentioned above, his curve barely reaches a threshold level of ξ = 2. Furthermore, the results for levels ξ = 2.5 and ξ = 3 are confirmed by Monte Carlo runs with sample sizes of 10.000 and 5.000.000, respectively. In this range the results match perfectly. Higher threshold levels could not be confirmed with reasonable computational effort.

3.3 SDOF system with sliding pendulum seismic isolation

The structural model considered here is represented by a single-degree-of-freedom oscillator. The structure is supported by a seismic protection device (a sliding isolation pendulum (SIP) system, see e.g. [9]). The mechanical model of the device together with the structure to be protected is shown in Fig. 10. This model contains a friction element with a maximum

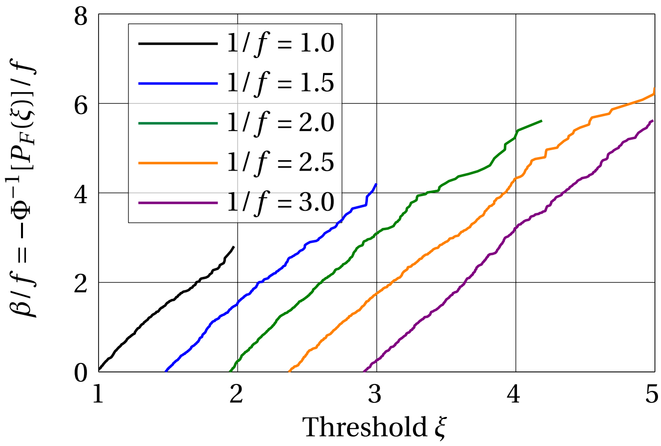


Figure 8: Distribution function of peak absolute response of SDOF system for different sampling scales f

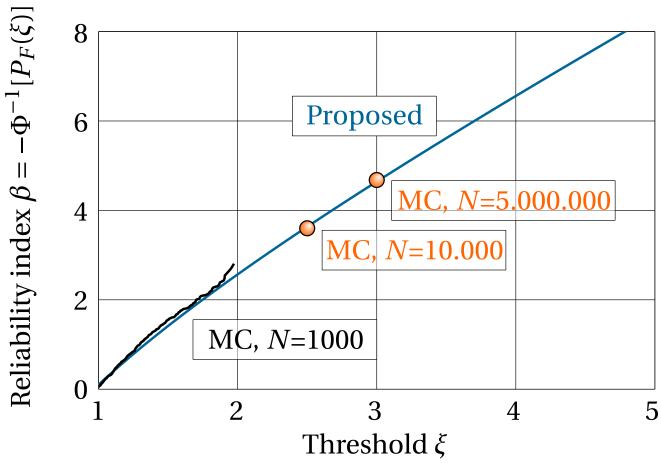


Figure 9: Exceedance probability of SDOF systems under white noise

transmissible force r which is computed from the weight G of the structure acting on the friction device and the friction coefficient μ. A re-centering spring due to the pendulum effect is included. Its spring constant can be computed from the weight of the structure and the effective radius of curvature R of the pendulum system.

The spring k represents the structural stiffness, the structural mass is given by m. The displacement of the structural mass is given by the variable x, the offset of the friction device by the variable z. In order to represent structural damping, a viscous damper c is added in parallel to the spring k. In the following computations, the structural damping is represented by a damping ratio $\zeta = 0.02$.

The earthquake excitation is introduced in terms of the ground acceleration $a(t)$ which is modeled as an amplitude-modulated white noise:

$$a(t) = w(t)e(t) \tag{17}$$

in which $w(t)$ is stationary white noise with spectral density $S_0 = 0.01$ m^2/s^3 and $e(t)$ is given as

$$e(t) = 4\left[\exp(-0.25t) - \exp(0.5t)\right] \tag{18}$$

The total time duration is $T = 25$ s, the time step chosen is $\Delta t = 0.05$ s.

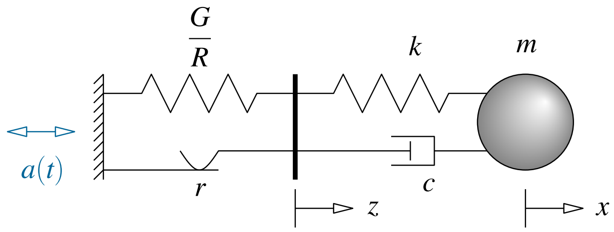


Figure 10: Mechanical model for structure and seismic protection device

For this system, the force F_r in the friction element can be computed from the equilibrium condition

$$F_r + \frac{G}{R} z = k(x - z) \rightarrow F_r = Kx - \left(k + \frac{G}{R}\right) z \tag{19}$$

If there is no slip, then the rate of change of this force due to the rates $\dot{x}$ and $\dot{z}$ is therefore

$$\dot{F}_r = K\dot{x} - \left(k + \frac{G}{R}\right) \dot{z} \tag{20}$$

Slip in the friction element occurs under one of the the following two conditions:

$$(F_r \geq r \wedge \dot{x} > 0) \vee (F_r \leq -r \wedge \dot{x} < 0) \tag{21}$$

If one of these conditions is met, then there is no increment in the force F_r, i.e. $\dot{F}_r = 0$ and therefore

$$\dot{z} = \frac{k}{k + \frac{G}{R}} \dot{x} \tag{22}$$

Otherwise we have $\dot{z} = 0$. Together with the state vector form of the equation of motion

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -k(x - z) - c(y - \dot{z}) + a(t) \end{aligned} \tag{23}$$

this results in a system of three first-order differential equations which can be solved e.g. by the Runge-Kutta method. In the numerical analysis the values for the structural parameters are m = 1 kg, k = 100 N/m, c = 0.1 Ns/m. The acceleration of gravity g = 9.81 m/s^2, and the friction coefficient $\mu = 0.04$.

The primary purpose of the seismic isolation lies in reducing the forces transmitted from the ground into the structure. This effect is easily quantified in terms of the maximum acceleration of the structural mass m. Fig. 11 compares the CDF of the peak acceleration for different values of the radius of curvature R of the sliding isolation pendulum system.

It can be seen that the probability of exceeding large acceleration values is dramatically reduced by the SIP. Since the compete tail of the first-passage distribution is obtained, reliability-based optimization of sliding pendulum systems as presented e.g. in [3] can be further improved.

3.4 Nonlinear MDOF systems

An MDOF system with Duffing type nonlinearity as presented in [10] is analyzed. This system is driven by an earthquake excitation $a(t)$ which is defined as the output of a 4$^{\text{th}}$ order

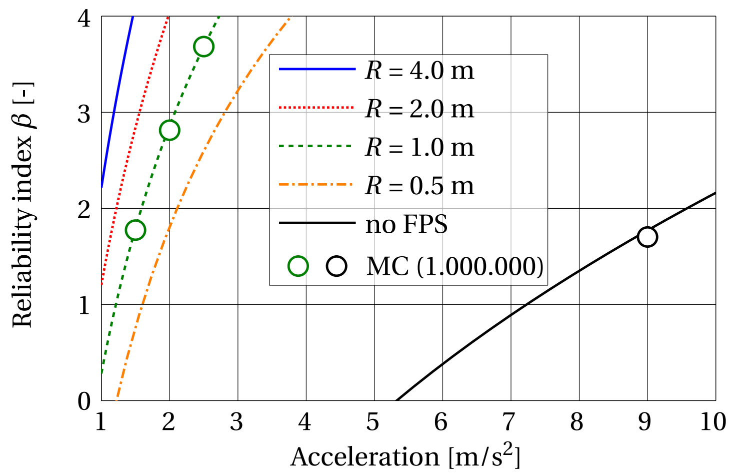


Figure 11: Tail of peak acceleration of structure depending on radius of curvature R for FPS

linear filter driven by non-stationary white noise $w(t)$. The noise is defined by its covariance function

$$E[w(t)w(t+\tau)] = I\delta(\tau)\cdot h(t);\; h(t) = \begin{cases} t/2 & , \; 0s \le t \le 2s \\ 1 & , \; 2s \le t \le 10s \\ \exp\big(-0.1(t-10)\big) & , \; t \ge 10s \end{cases} \tag{24}$$

with $I = 0.08$ m^2/s^3. The linear filter is given by the equations

$$\dot{\mathbf{X}}(t) = \mathbf{A}_{EQ}\mathbf{X}(t) + \mathbf{B}_{EQ}w(t);\; a(t) = \mathbf{C}_{EQ}\mathbf{X}(t) \tag{25}$$

$$\mathbf{A}_{EQ} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\Omega_{1g}^2 & -2\zeta_{1g}\Omega_{1g} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ \Omega_{1g}^2 & 2\zeta_{1g}\Omega_{1g} & -\Omega_{2g}^2 & -2\zeta_{2g}\Omega_{2g} \end{bmatrix} \quad \mathbf{B}_{EQ} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \tag{26}$$

$$\mathbf{C}_{EQ} = \begin{bmatrix} \Omega_{1g}^2 & 2\zeta_{1g}\Omega_{1g} & -\Omega_{2g}^2 & -2\zeta_{2g}\Omega_{2g} \end{bmatrix} \tag{27}$$

Numerical values chosen are Ω_{1g} =15 rad/s, ω_{2g} = 0.3 rad/s, ζ_{1g} = 0.8 and ζ_{2g} = 0.995. The MDOF system is governed by the equations

$$\mathbf{M}\ddot{\mathbf{u}}(t) + \mathbf{C}\dot{\mathbf{u}}(t) + \mathbf{K}\big(\mathbf{u}(t)\big)\mathbf{u}(t) = \mathbf{F}(t) = \mathbf{m}a(t) \tag{28}$$

in which the matrices are given as

$$\mathbf{M} = \begin{bmatrix} m_1 & 0 & \dots & 0 \\ 0 & m_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & m_{10} \end{bmatrix}; \quad \mathbf{m} = \begin{bmatrix} m_1 \\ m_2 \\ \vdots \\ m_{10} \end{bmatrix} \tag{29}$$

$$\mathbf{C} = \begin{bmatrix} c_1 + c_2 & -c_2 & \dots & 0 \\ -c_2 & c_2 + c_3 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & c_{10} \end{bmatrix} \mathbf{K} = \begin{bmatrix} \bar{k}_1 + \bar{k}_2 & -\bar{k}_2 & \dots & 0 \\ -\bar{k}_2 & \bar{k}_2 + \bar{k}_3 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \bar{k}_{10} \end{bmatrix} \tag{30}$$

$$\bar{k}_i = k_i \left(1 + \epsilon \left(\frac{u_i(t) - u_{i-1}(t)}{\delta u_{ref}}\right)^2\right); i = 1, \cdots, 10 \tag{31}$$

The system parameters are chosen as $\delta u_{ref} = 0.02$ m, $\epsilon = 0.1$, $m_1 = \cdots = m_{10} = 10.000$ kg, $k_1 = k_2 = k_3 = 40$ MN/m, $k_4 = k_5 = k_6 = 36$ MN/m, $k_7 = k_8 = k_9 = k_{10} = 32$ MN/m, $c_i = 2\zeta_i\sqrt{m_i k_i}$ and $\zeta_i = 0.04$ for $i = 1, \cdots, 10$. The time duration is $T = 20$ s and the time step is chosen as $\Delta t = 0.005$ s. Therefore the load discretization requires 4000 random variables. The failure criterion is the relative displacement between stories 9 and 10 exceeding a threshold value ξ. Here a range of threshold values from 0.01 m to 0.05 m is considered. The asymptotic sampling procedure is carried out with 5 runs with 1000 simulations each. The scaling factor f is thereby varied between 0.33 and 1.0.

The resulting reliability index for the first passage probability depending on the value of the threshold level ξ is shown in Fig. 12.

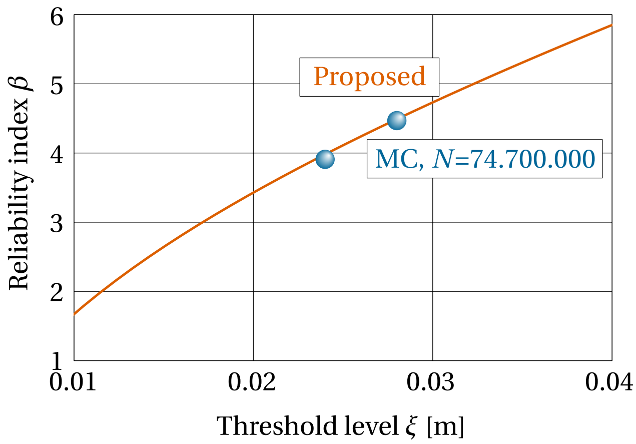


Figure 12: First passage probability of nonlinear MDOF system under non stationary colored noise. Monte Carlo results are taken from [10].

4 CONCLUSIONS

In view of the results as obtained it may be concluded that the suggested Monte Carlo based method using Asymptotic Sampling to compute the tail probabilities of stochastic dynamic response quantities performs remarkably well. The method provides asymptotically unbiased estimated for the tails of the distribution function. The examples as analyzed here also show that the treatment several thousands of random variable does not pose any difficulties. This very useful property is due to the sole use of Monte Carlo sampling without any specific assumptions such as location of the design point or similar. Therefore the method as presented is particularly suitable for the analysis of high-dimensional problems such as the first-passage in random vibrations. For this class of problems it can be seen that:

- The results agree very well with accurate results in the probability range which is verifiable by Monte Carlo simulation.
- The proposed method can readily predict extremely small failure probabilities with substantially reduced computational effort as compared to conventional MC-based approaches.

Due to its efficacy, the method is especially well-suited for computationally demanding problems such as reliability-based structural optimization.

ACKNOWLEDGMENT

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