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A POSSIBILISTIC APPROACH FOR LINEAR ISOTROPIC ELASTICITY USING THE FUZZY FINITE ELEMENT METHOD

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Abstract. *The uncertainty quantification has become an inalienable factor in many physical and engineering applications for determination of reliable results with the finite element method (FEM). In this work we investigate the uncertainties characterized by imprecise probabilities [12, 13]. To this end, a membership function is interpreted as a possibility distribution and a possibility distribution as a family of probability distributions. In the basic preliminaries we summarize the fundamentals of the fuzzy set theory as well as the possibility theory including a probability-possibility transformation [11]. Moreover, a linear elastic body with two fuzzy input material parameters is studied. The main objectives are twofold: Firstly, determination of two unknown fuzzy parameters and secondly, numerical computation of the system response under consideration of the interaction between fuzzy parameters using the possibilistic evaluation of the fuzzy finite element method (FFEM). The α-level discretization technique [2] is applied in order to reduce the fuzzy arithmetic based FEM to an interval arithmetic based FEM. Finally, our method is applied in a numerical example for a plate with a ring hole.*

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1 INTRODUCTION

In modern engineering sciences the uncertainty quantification as well as its suitable application in simulations with the finite element method (FEM) becomes essential in order to obtain realistic results. In this context, the uncertainty arises from linguistic, informal or statistical properties and may be attached to material parameters, geometry parameters and/or loading parameters. Different causes of the uncertainty require different characterization techniques. Thus, statistical (or stochastic) based uncertainty is described by randomness [2], i.e. random variables and/or random fields. This kind of uncertainty is referred to as the *aleatoric* uncertainty, whereas the other kinds of uncertainties (especially the one based on incomplete information) are assigned to the *epistemic* type [4]. To this end, the deterministic finite element method is expanded to the stochastic FEM (SFEM) [5, 14, 15] in order to handle the aleatoric uncertainty. Accordingly, there exist the interval FEM and the fuzzy FEM (FFEM) [2, 16] in order to comprise the epistemic uncertainty in the simulation. Naturally, in the engineering sciences a pure aleatoric model is not able to exist, since the determination of an *exact* probability distribution can only be achieved in the infinite trial. For this reason, every uncertain model is (partly) of epistemic nature. The combination of both types of uncertainty is referred to as *imprecise probability* [12, 13]. For this, the probability-box FEM (P-box FEM) [17] and/or fuzzy stochastic FEM (FSFEM) [2, 18] are applied in order to consider imprecise probabilities in the FEM.

Furthermore, the *possibility theory* first introduced by Zadeh [19] and expanded by Dubois and Prade [6, 13] was formulated in order to deal with imprecise probabilities. To this end, membership functions of the fuzzy set theory [2, 6] may be interpreted as possibility distributions where one of them comprises a family of probability distributions [11]. A main application area of the possibility theory is the field of computer science or the cognitive psychology as can be seen e.g. in [20].

This work describes the possibilistic approach in the finite element method based on incomplete information for material parameters referred to as *design variables*. The incompleteness is justified in a small number of experimental observations. Thus, the underlying probability distribution cannot be specified in contrast to a family of *possible* probability distributions. That is, we use the possibilistic interpretation of the fuzzy finite element method in order to describe imprecise probabilities caused by fuzzy material parameters.

An outline of this work is as follows: Section 2 summarizes the basic equations of the fuzzy set theory and the related possibility theory including the *possibility distribution* and *possibility measure*, respectively. Furthermore, the continuous variational formulation for a linear elastic body with fuzzy material parameters are introduced. In Section 3 the spatial discretization is provided in order to obtain the linear elastic fuzzy equation system. Finally, Section 4 consolidates the previous explanations in the representative example for a two parameter model of isotropic linear elasticity under consideration of the interaction of fuzzy material parameters.

2 BASIC PRELIMINARIES

The aim of this work is a possibilistic evaluation of the (linear elastic) fuzzy finite element method based on fuzzy material parameters. In this context, the *fuzziness* arises from a small number of experimental observations. This section recaps the essential definitions.

2.1 Fuzzy sets

Based on [1, 2, 3] we define for a design space $S \subset \mathbb{R}^{n_s}$ with n_s as the number of input (design) variables s_i according to $\underline{s} = (s_1,\ s_2, \ldots, s_{n_s})$

$$\hat{S} = \{(\underline{s}, \mu_S(\underline{s})) \mid \underline{s} \in \mathbb{R}^{n_s},\ \mu_S(\underline{s}) = \min[\mu_{S_i}(s_i)]\}, \tag{1}$$

as the n_s dimensional fuzzy set $\hat{S}$ with the n_s dimensional membership function μ_S, where

$$\hat{S}_i = \{(s_i, \mu_{S_i}(s_i)) \mid s_i \in \mathbb{R},\ \mu_{S_i} \in [0,1]\} \tag{2}$$

is the one dimensional fuzzy set for $S_i \subset \mathbb{R}$ with the one dimensional membership function μ_{S_i}.

2.2 Fuzzy arithmetic

In order to perform mathematical operations with fuzzy sets the α-level discretization technique is used [2, 11]. For this, the α_k-cut of a fuzzy set $\hat{S}$ is defined as the set

$$S_{\alpha_k} = \{\underline{s} \mid \mu_S(\underline{s}) \geq \alpha_k,\ \underline{s} \in \mathbb{R}^{n_s}\} = S_{1,\alpha_k} \times S_{2,\alpha_k} \times \cdots \times S_{n_s,\alpha_k} \subset \mathbb{R}^{n_s}, \tag{3}$$

where $S_{i,\alpha_k} := [s^L_{i,\alpha_k}, s^R_{i,\alpha_k}]$ for $k \in \mathbb{N}_0$ and $i = 1, \ldots, n_s$. This approach allows the usage of an ordinary interval arithmetic technique [7, 21] in order to perform mathematical operations with fuzzy sets. The higher the number of cuts, the more accurate is result, however, it becomes numerically more demanding. Figure 1 illustrates the α-level discretization on a one dimensional trapezoidal fuzzy set $\hat{S}_i$.

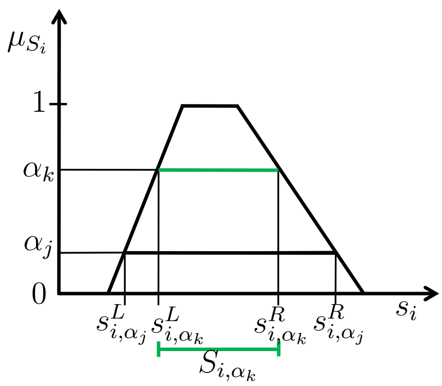

Figure 1: α-cuts of a fuzzy trapezoidal number $\hat{S}_i$

Note that, the inclusion of the *interaction* (which is defined as being the mutual dependency of fuzzy variables [2]) is important in order to obtain most realistic results with the interval arithmetic. Section 4 illustrates the meaning of interaction in the representative example.

2.3 Possibility vs. probability

There are several interpretations of fuzziness (see e.g. [22]). Initially, fuzzy sets were introduced to model vague linguistic knowledge. There, the membership degree represents a degree of similarity of preference, acceptability, suitability or (more general) a degree of truth. A further interpretation is to consider a fuzzy set as a set of probability measure [6, 13, 20]. To this end, membership functions are interpreted as possibility distributions and a possibility distribution as a family of probability distributions. Thus, in the engineering sciences, the terminology of *possibility theory* allows the examination of *imprecise probabilities*, whereas the ordinary fuzzy set theory is used for the epistemic uncertainty.

2.3.1 Possibility distribution as a family of probability distributions

According to [11] a possibility distribution is a function

$$\pi\colon S \longrightarrow [0,1] \subseteq \mathbb{R}, \tag{4}$$

with S as the design space of design variables. Following [11] we formulate the equations

$$\begin{aligned} &1. \quad \mathcal{P}(\pi) = \{p \in \mathcal{P},\ \forall A \subset \mathbb{R}^{n_s},\ N(A) \le P(A) \le \Pi(A)\}, \text{ with} \\ &2. \quad \Pi(A) = \sup_{\underline{s}\in A} \pi(\underline{s}) \quad \text{as the possibility measure and} \\ &3. \quad N(A) = 1 - \Pi(\bar{A}) \quad \text{as the necessity measure,} \end{aligned} \tag{5}$$

where $\mathcal{P}$ is the family of all possible probability distributions and P is the probability measure associated with the probability distribution p. From Eq. (4) and Eq. (5) we observe, that the possibility distribution π contains all probability distributions p that are upper bounded by the possibility measure Π, whereas the *necessity* measure N may be interpreted as the lower bound. For more information about the possibility measure and the necessity measure see e.g. [6, 23]. In this work, we restrict our examination on the *possibility*. We use this term as an abbreviation for the often (synonymically) used terms *possibility distribution*, *possibility measure* and/or *possibility degree*.

2.3.2 Probability-possibility transformation for limited experimental data

As already indicated our attempt is the inclusion of sparse experimental data in a simulation as well as the interpretation of the corresponding system response. That is, we have to consider especially non-available information in order to obtain most realistic results based on sparse information. The determination of the *exact* probability distribution is not possible in case of sparse experimental data. Nevertheless, we may embrace *the family* of possible probability distributions. The idea is to transform the (possible) marginal probability density functions in two possibility distributions. A subsequent combination to *one* possibility distribution encoding the family of all possible probability distributions which may arise considering the sparse experimental information.

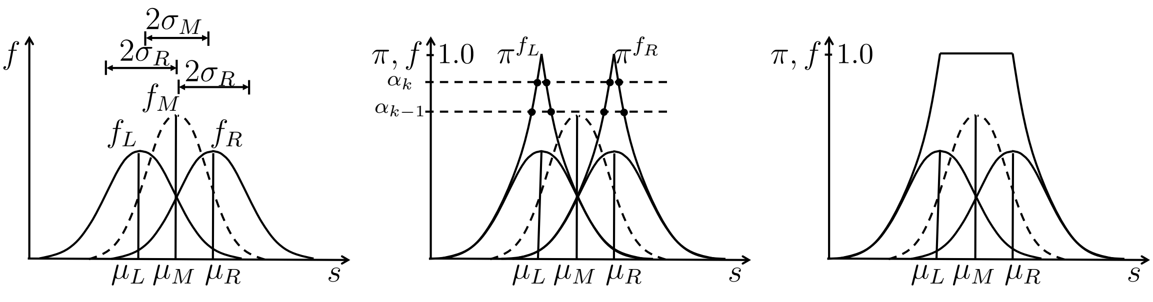

Figure 2: Schematic representation of a) marginal density functions f_L and f_R with a *mean* density function f_M, b) corresponding possibility distributions π^{f_L} and π^{f_R} and c) the possibility distribution π encoding all possible normal density functions

We describe the procedure considering the schematic illustration in Fig. 2. Let f_M be a (normal) probability density function of the variable s (as can be seen in Fig. 2.a) with corresponding expected value μ_M and standard deviation σ_M calculated from n sparse experimental data of s. Although it is unlikely that the density function f_M describes the *real* density function of the design variable s, nevertheless, it is a possible one. With a specific confidence level (of e.g. 95%) one obtains the confidence intervals $\mu = [\mu_L,\ \mu_R]$ for possible expected values with $\mu_L \leq \mu_M \leq \mu_R$ and $\sigma = [\sigma_L,\ \sigma_R]$ for possible standard deviations with $\sigma_L \leq \sigma_M \leq \sigma_R$ considering the n experimental data (see [8, 9] for more information about confidence intervals). Thus, we may establish the marginal (normal) density function f_L with expected value μ_L and standard deviation σ_R and the marginal (normal) density function f_R with expected value μ_R and standard deviation σ_R as illustrated in Fig. 2.a. Note that, both marginal density functions have the largest standard deviation σ_R. Consider for exemplification $f := f_L$ as the standard normal density function, i.e.

$$f = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}s^2\right), \tag{6}$$

with $\mu_L = 0$ and $\sigma_L = 1$. The probability-possibility transformation

$$T\colon \begin{cases} D \longrightarrow \mathrm{P} \\ f \longmapsto \pi^f \end{cases} \tag{7}$$

assigns to the density function $f \in D$ the *maximum* specific possibility distribution π^f, where D is a set of density functions and P is a set of possibility distributions. In order to obtain the maximum specific possibility distribution with the relation in Eq. (7) one has to determine the confidence intervals [11]

$$\mathrm{I}_{\beta_k} = \{s \mid s \in f^{-1}[d], \forall d \in [c_k, +inf)\}, \tag{8}$$

with

$$\beta_k = \int_{\{s|f(s)\geq c_k\}} f(s)ds, \tag{9}$$

for $k = 1, \ldots, n_\alpha$ and $c_k > 0$ where n_α is the number of alpha cuts. Then, considering the relation (derived in [11])

$$S_{\alpha_k} = [s^L_{\alpha_k}, s^R_{\alpha_k}] := \mathrm{S}_{1-\beta_k} = \mathrm{I}_{\beta_k} \tag{10}$$

one is able to determine the α-cuts of the corresponding maximum specific possibility distribution. Figure 3 illustrates the transformation of the standard normal density function f (shown in Fig. 3.a) to the maximum specific possibility distribution π^f (shown in Fig. 3.b).

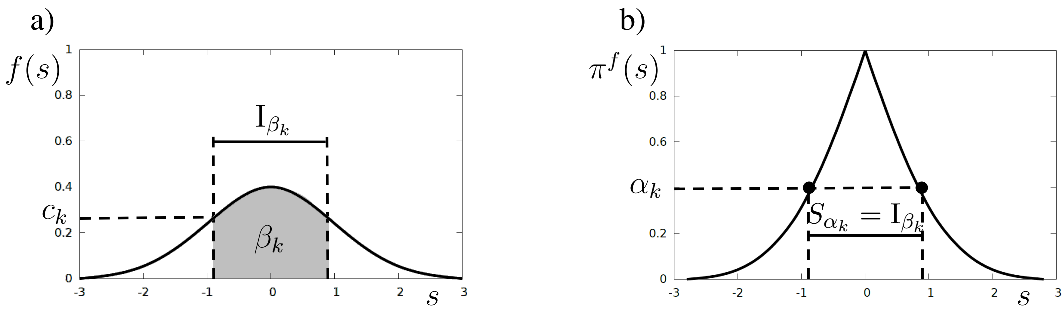

Figure 3: Probability-possibility transformation: a) Density function of a standard normal distribution, b) corresponding maximum specific possibility distribution

Furthermore, Fig. 2.b. illustrates schematically the transformation of marginal density functions f_L and f_R to the corresponding possibility distributions π^{f_L} and π^{f_R} using Eq. (7) - Eq. (10). Subsequently, we formulate the (common) maximum specific possibility distribution π as

$$\pi(s) := \begin{cases} \pi^{f_L}(s) & \text{for} \quad s < (\pi^{f_L})^{-1}(1) \\ \pi^{f_R}(s) & \text{for} \quad s > (\pi^{f_R})^{-1}(1) \\ 1 & \text{for} \quad s \in [(\pi^{f_L})^{-1}(1), (\pi^{f_R})^{-1}(1)] \end{cases} \tag{11}$$

Thus, the possibility distribution π in Eq. (11) encodes all possible probability density functions which may arise considering the n sparse experimental data. Eventually, π constitutes the input function of the design variable s for the finite element method as can be seen in the representative example in Section 4.

2.4 Governing equations for an elastic body

The considered linear elastic material behavior is characterized by fuzzy material parameters. Building on deterministic governing equations (see e.g. [10]) on the region Ω occupied by the elastic body, the following variational formulation is constituted:

$$\int_{\Omega} \delta\hat{\boldsymbol{\varepsilon}} : \hat{\mathbb{C}} : \hat{\boldsymbol{\varepsilon}} \, d\Omega = \int_{\partial\Omega_\sigma} \bar{\boldsymbol{t}} \, \delta\hat{\boldsymbol{u}} \, d\partial\Omega. \tag{12}$$

In Eq. (12) we have $\hat{\mathbb{C}}$ as the fuzzy elasticity tensor, $\delta\hat{\boldsymbol{u}}$ as the fuzzy virtual displacement, $\hat{\boldsymbol{\varepsilon}}$ as the fuzzy strain tensor and $\delta\hat{\boldsymbol{\varepsilon}}$ as the fuzzy virtual strain tensor with corresponding Dirichlet and Neumann boundary conditions

$$\boldsymbol{u} = \bar{\boldsymbol{u}} \text{ on } \partial\Omega_u, \quad \bar{\boldsymbol{t}} = \boldsymbol{\sigma} \cdot \boldsymbol{n} \text{ on } \partial\Omega_\sigma \tag{13}$$

with non-intersecting boundaries $\partial\Omega = \partial\Omega_\sigma \cup \partial\Omega_u$ and $\emptyset = \partial\Omega_\sigma \cap \partial\Omega_u$.

3 NUMERICAL IMPLEMENTATION

3.1 Spatial discretization with fuzzy material parameters

We consider a linear elastic isotropic material described by fuzzy input material parameters Young's modulus $\hat{E}$ and Poisson's ration $\hat{\nu}$. Using the displacement and strain approximation

$$\begin{aligned} \underline{\hat{u}} &\approx \sum_{i=1}^{n_k} N_i \, \underline{\hat{u}}_i^* = \underline{N} \, \underline{\hat{u}}^* \quad \text{and} \\ \underline{\hat{\varepsilon}} &\approx \underline{B} \, \underline{\hat{u}}^*, \end{aligned} \tag{14}$$

with $\underline{B}$ as the associated derivative matrix of shape functions N_i, $\underline{\hat{u}}^*$ as the nodal displacement vector and n_k as the number of nodes per element, the following fuzzy equation system is derived:

$$\underline{\hat{K}} \underline{\hat{u}}^* = \mathop{\mathsf{A}}_{e=1}^{n_e} \int_{^eR} \underline{B}^T \, \underline{\hat{C}} \, \underline{B} \, dR \underline{\hat{u}}^* = \underline{f} = \mathop{\mathsf{A}}_{e=1}^{n_e} \int_{\partial^eR} \underline{\bar{t}} \, \underline{N} \, d\partial R \tag{15}$$

In Eq. (15) A represents the assembly operator, ${}^eR$ is the domain of one finite element and n_e is the number of elements. Furthermore, the fuzzy elasticity matrix is given as

$$\underline{\hat{C}} = \underline{\hat{C}}_{dev} + \underline{\hat{C}}_{vol} = 2\hat{G} \underline{I}_{dev}^C + \hat{K} \underline{m} \underline{m}^T \tag{16}$$

with

$$\hat{G} = \frac{\hat{E}}{2(1+\hat{\nu})} \quad \text{fuzzy shear modulus,}$$
$$\hat{K} = \frac{\hat{E}}{3(1-2\hat{\nu})} \quad \text{fuzzy bulk modulus} \tag{17}$$

and

$$\underline{I}^C_{dev} = \underline{I}^C - \frac{1}{3}\underline{m}\underline{m}^T, \quad \underline{I}^C = \begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & \frac{1}{2} & & \\ & & & & \frac{1}{2} & \\ & & & & & \frac{1}{2} \end{bmatrix}, \quad \underline{m} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{18}$$

The shear modulus and the bulk modulus in Eq. (17) become fuzzy, since they depend on fuzzy material parameters $\hat{E}$ and $\hat{\nu}$. Therefore, the elasticity matrix in Eq. (16) is also fuzzy. For arithmetic operations with fuzzy variables in Eq. (16) and Eq. (17) the interval arithmetic technique is used considering the interaction of fuzzy variables.

3.2 Possibility determination of the system response

The possibility of the system response $\underline{\hat{u}}^*$ in Eq. (15) is constituted as the possibility of a quantity of interest $\pi(Q(\underline{\hat{u}}^*))$. In this context, the quantity of interest may be the displacement at certain regions or the displacement of a single node for example. For more information about quantity of interests see, e.g. [26, 25, 27]. The possibility is calculated by determining the minimum and the maximum of the quantity of interest $Q(\underline{\hat{u}}^*)^{\alpha_k}$ for $k = 1, \ldots, n_\alpha$ using the interval arithmetic technique.

4 REPRESENTATIVE EXAMPLE

Figure 4 illustrates the geometry, the loading and the FE-discretization of a plate with a ring hole with linear triangular elements under plain strain condition. Our goal is the computation of the possibility for the displacement $\hat{u}^*_A$ of node A (i.e. $\pi_A := \pi(Q(\underline{\hat{u}}^*) = \hat{u}^*_A))$ illustrated in Fig. 4.b.

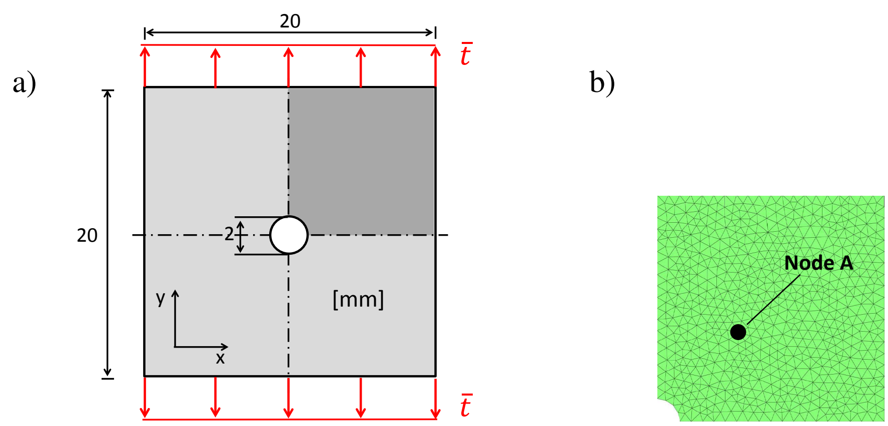

Figure 4: Plate with a ring hole: a) Geometry and b) discretization with triangular elements

4.1 Possibility establishment for the input parameters

Figure 5 shows a histogram of experimental results for Young's modulus E and Poisson's ratio ν for an adhesive material. Experimental observations of this material are published in [24] for combined tension torsion tests. Considering this sparse information we use the probability-possibility transformation explained in Section 2.3.2 for both material parameters in order to obtain their possibilities as illustrated in Figure 6. The corresponding possibilities for the fuzzy shear modulus and the fuzzy bulk modulus from Eq. (17). The interaction domain with the joint possibility $\pi(\underline{s} = (G, K)) = 0.5$ for $\alpha_6 = 0.5$ is illustrated in Figure 7.

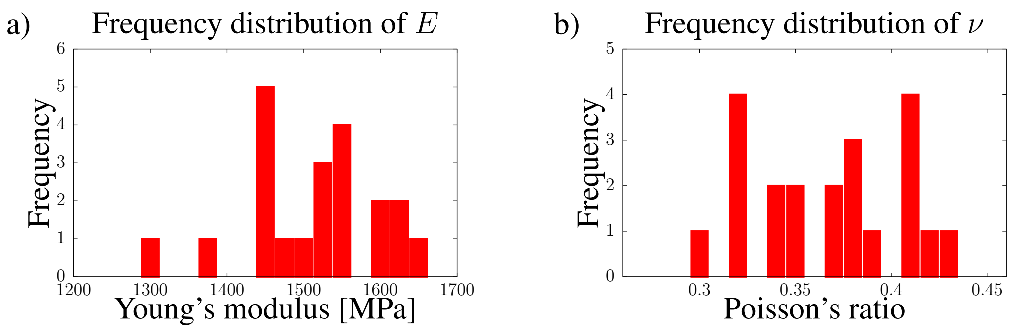

Figure 5: Experimental results for a) Young's modulus and b) Poisson's ratio

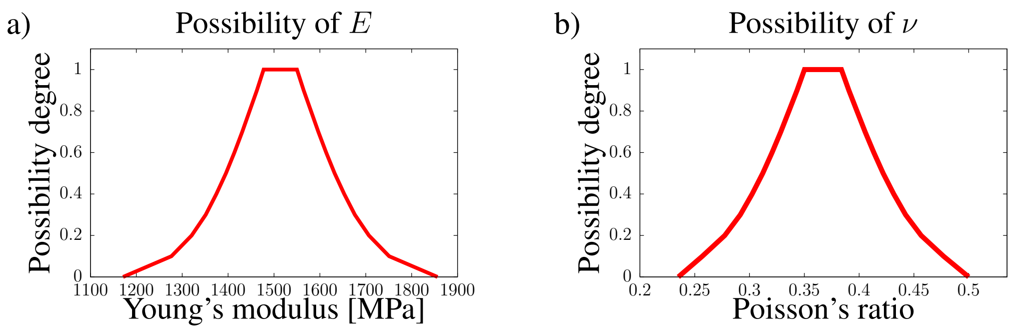

Figure 6: Possibility distribution of a) Young's modulus and b) Poisson's ratio

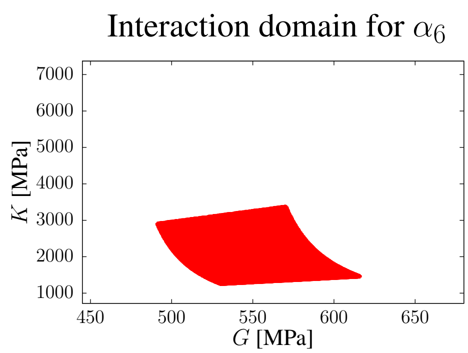

Figure 7: Interaction domain for the α_6-cut of the shear modulus and the bulk modulus

4.2 Possibility of the displacement for node A

Considering the interaction of the shear modulus and the bulk modulus in Eq. 17 we use the α-level discretization technique in order to calculate π_A. In this context, Fig. 7 illustrates the α_6-cut of the joint possibility distribution of the shear modulus and the bulk modulus. By determining the minimum and the maximum of $\hat{u}_A^*$ in every α-cut the possibility π_A is calculated. In this context, due to the monotonicity of $\hat{u}_A^*(G, K)$ the solutions for the minimum and maximum (in every α-cut) are in the corners of the interaction domain of the shear modulus and the bulk modulus. Figure 8 illustrates the possibility distribution π_A of the displacement $\hat{u}_A^* := (u_{A,x},\ u_{A,y})$ for node A calculated with 11 α-cuts (with equidistant distances of 0.1) under the assumption of independency for the x and y directions. In order to establish this *two* dimensional representation for the possibility π_A of the displacement $\hat{u}_A^*$, the principle of Eq. (1) is used.

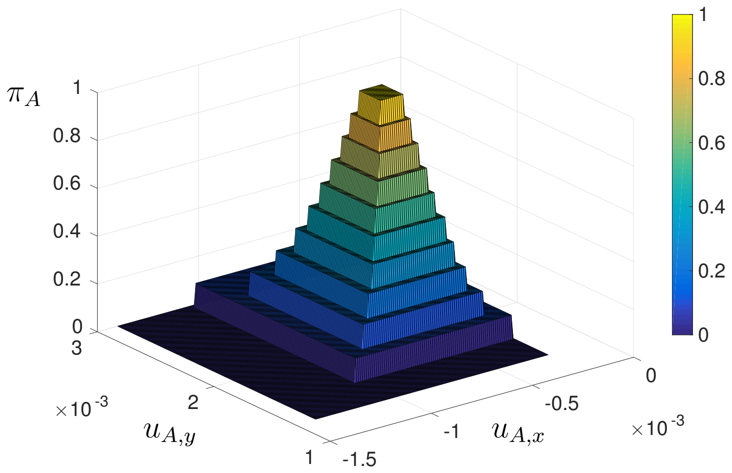

Figure 8: Possibility of the displacement for node A in Figure 4

5 CONCLUSION

Different kinds of uncertainty require different characterization techniques. This work presents a possibilistic evaluation of the linear elastic fuzzy finite element method with fuzzy input parameters based on sparse experimental data. To this end, possible marginal probability density functions of the fuzzy input parameters are transformed into possibility distributions using a probability-possibility transformation. Subsequently, a collective possibility distribution encoding a family of possible density functions is established. The possibility distributions of the material parameters are implemented as input functions in the fuzzy finite element method. In order to perform mathematical operations with possibility distributions respectively membership functions the α-level discretization technique is used. It allows the calculation with an interval arithmetic based FEM. To ensure realistic results it is necessary to incorporate the interaction of fuzzy variables while using interval arithmetic. The possibilistic evaluation allows the examination of the uncertainty characterized by *imprecise probabilities* and may be interpreted as an upper bound of the probabilities.

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