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Eccomas Proceedia UNCECOMP (2017) 289-299

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UNCECOMP 2017
2nd ECCOMAS Thematic Conference on
Uncertainty Quantification in Computational Sciences and Engineering
M. Papadrakakis, V. Papadopoulos, G. Stefanou (eds.)
Rhodes Island, Greece, 15–17 June 2017

INTERVAL DYNAMIC RESPONSE OF DISCRETIZED STRUCTURES WITH UNCERTAIN-BUT-BOUNDED PARAMETERS

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Keywords: Uncertain-but-bounded parameters, Interval Analysis, Vibration Analysis, Structural Dynamic Response, Lower and Upper Bounds of time-history responses.

Abstract. *The dynamic analysis of structures plays an important role in the design of structural systems. Unfortunately, mechanical properties are usually uncertain due to physical imperfections and model inaccuracies. Therefore, it is important to estimate the effects of these uncertainties on the structural dynamic response. Since information regarding structural parameters is often quite limited, the use of non-probabilistic approaches is deemed advisable to realistically characterize uncertainties. In the framework of these approaches, the interval method seems today the most widely adopted.*

The interval dynamic analysis of structures with slight parameter fluctuations is usually performed by applying the Interval Perturbation Method (IPM). However, the effectiveness of the IPM is limited to small uncertainties since the effect of neglecting higher-order terms is unpredictable.

In this paper, a novel procedure, able to overcome the main limitations of the IPM is proposed for evaluating the lower and upper bounds of the response of linear structural systems, with uncertain-but-bounded properties, subjected to dynamic deterministic excitations. The accuracy of the proposed method is assessed by evaluating the bounds of the response of a truss structure and a shear type frame subjected to seismic acceleration.

Peer-review under responsibility of the organizing committee of UNCECOMP 2017.
doi: 10.7712/120217.5370.16883

1 INTRODUCTION

The dynamic analysis of structures plays an important role in the design of structural systems. Unfortunately, the structural mechanical properties are usually uncertain due to physical imperfections and model inaccuracies. Therefore, it is important to estimate the effects of these uncertainties on the structural dynamic response. However, while the numerous available data permit to model with good accuracy the excitations as stochastic processes, unfortunately the data about structural parameters are often quite limited. It follows that the probabilistic approach cannot always be realistically applied to represent structural uncertainties; indeed, it requires a wealth of data, often unavailable, to define the probability density function of the fluctuating structural parameters. Non-probabilistic approaches can be alternatively used to treat these uncertainties. In this framework, the interval model seems today the most suitable analytical tool [1, 2]. The main advantage of the interval model is that it provides analytically rigorous enclosures of the solution, but its application to practical engineering problems is not an easy task due to two main drawbacks commonly faced in the development of interval-based procedures for structural analysis: i) the drastic overestimation of the interval solution range due to the so-called *dependency phenomenon* [3]; ii) the high computational costs when the exact combinatorial approach, known as Vertex Method, is adopted [3, 4].

Based on the matrix perturbation theory and interval extension of functions, the upper and lower bounds of the dynamic response were obtained by Chen et al. [5] using Taylor series expansion. Subsequently, the *Interval Perturbation Method* (*IPM*), which is based on Taylor series expansion and parameter perturbation, has been introduced to evaluate the interval dynamic response of structures subjected to deterministic [6-9] or stochastic excitations [10]. More recently, Gao et al. [11] presented the *interval factor method* to calculate the dynamic response of truss structures. Yang et al. [12] proposed an interval analysis method for dynamical systems using Laplace transform to solve the equations of motion. Xia and Yu [13-15] developed a modified *IPM* based on the modified Neumann expansion for the response analysis of interval structures and interval structural-acoustic systems.

Although other methods are more accurate, the *IPM* or, equivalently, the First-Order Interval Taylor Series Expansion, is the most widely used to evaluate the interval structural dynamic response. The main advantages of the *IPM* are the flexibility and the simplicity of the mathematical formulation. However, since the effect of neglecting higher-order terms is unpredictable, the effectiveness of this method is limited to uncertainties with small intervals.

In this paper, a novel procedure, which exhibits the same advantages of the *IPM*, is presented. The proposed method evaluates the bounds of the interval structural dynamic response by using the same numerical procedures traditionally adopted in the dynamics of structural systems without uncertainties. Moreover, the drawbacks of the *IPM* are overcome, as shown in the Numerical Application section, where the lower and upper bounds of the dynamic response of a truss structure and a shear-type frame are evaluated.

2 PROBLEM STATEMENT

Without loss of generality, the attention is herein focused on a quiescent $n-$DOF classically damped linear structural system subjected to a deterministic excitation $\mathbf{f}(t)$. The equations of motion governing the dynamic response can be cast in the form:

$$\mathbf{M}\,\ddot{\mathbf{u}}(t)+\mathbf{C}\,\dot{\mathbf{u}}(t)+\mathbf{K}\,\mathbf{u}(t)=\mathbf{f}(t) \qquad (1)$$

where $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ are $n\times n$ mass, damping and stiffness matrices, respectively; $\mathbf{u}(t)$ and $\mathbf{f}(t)$ are $n\times 1$ deterministic vectors listing the nodal displacements and the external loads, respectively; finally a dot over a variable denotes differentiation with respect to time t. In the following, the Rayleigh model is adopted for the damping matrix.

The elements of the stiffness matrix are assumed to be affected by uncertainties which are described by the dimensionless uncertain-but-bounded parameters $\alpha_i^I \ (i = 1, 2, \ldots, r)$ with the apex I meaning interval variable. In fact, a realistic situation is herein considered in which available information on the structural parameters is not enough to justify an assumption on their probabilistic distribution. According to the *classical interval analysis* [16,17], the r uncertain structural parameters $\alpha_i^I \ (i = 1, 2, \ldots, r)$, introduced before, are assumed to be independent. By applying the interval algebra formalism, the $i-$th uncertain parameter can be defined as $\alpha_i^I \triangleq [\underline{\alpha}_i, \overline{\alpha}_i] \in \mathbb{IR}$, where $\mathbb{IR}$ denotes the set of all closed real interval numbers, while $\underline{\alpha}_i$ and $\overline{\alpha}_i$ are the lower bound (LB) and upper bound (UB), respectively. Let the uncertain parameters α_i^I be collected into the interval vector $\boldsymbol{\alpha}^I = [\alpha_1^I, \ \alpha_2^I, \ \ldots, \ \alpha_r^I]^{\mathrm{T}}$, with the apex T meaning transpose operator, which is a bounded set-interval vector of real numbers $\boldsymbol{\alpha}^I \triangleq [\underline{\boldsymbol{\alpha}}, \overline{\boldsymbol{\alpha}}] \in \mathbb{IR}^r$, such that $\underline{\boldsymbol{\alpha}} \leq \boldsymbol{\alpha} \leq \overline{\boldsymbol{\alpha}}$, with the symbols $\underline{\boldsymbol{\alpha}}$ and $\overline{\boldsymbol{\alpha}}$ denoting the LB and UB vectors.

By taking into account the structural mechanical uncertainties, Eq.(1) can be rewritten as:

$$\mathbf{M}\,\ddot{\mathbf{u}}(\boldsymbol{\alpha}, t) + \mathbf{C}(\boldsymbol{\alpha})\,\dot{\mathbf{u}}(\boldsymbol{\alpha}, t) + \mathbf{K}(\boldsymbol{\alpha})\,\mathbf{u}(\boldsymbol{\alpha}, t) = \mathbf{f}(t), \qquad \boldsymbol{\alpha} \in \boldsymbol{\alpha}^I = [\underline{\boldsymbol{\alpha}}, \overline{\boldsymbol{\alpha}}] \tag{2}$$

where the stiffness matrix $\mathbf{K}(\boldsymbol{\alpha}^I)$ as well as the displacement vector $\mathbf{u}(\boldsymbol{\alpha}^I, t)$ depend on the interval parameters α_i^I collected into the vector $\boldsymbol{\alpha}^I$. Moreover, since the Rayleigh model is adopted for the damping matrix, the following relationship holds:

$$\mathbf{C}(\boldsymbol{\alpha}^I) = c_M\,\mathbf{M} + c_K\,\mathbf{K}(\boldsymbol{\alpha}^I), \tag{3}$$

where c_M and c_K are the Rayleigh damping constants having units s^{-1} and s, respectively.

According to the *improved interval analysis* [10], the *i*-th real interval variable α_i^I can be written in the following *affine form* [18]:

$$\alpha_i^I = \alpha_{0,i} + \Delta\alpha_i\,\hat{e}_i^I, \quad (i = 1, 2, \ldots, r) \tag{4}$$

where $\hat{e}_i^I \triangleq [-1, 1]$ is the so-called *Extra Unitary Interval* (*EUI*) [10]; $\alpha_{0,i}$ is the midpoint value (or mean) and $\Delta\alpha_i$ denotes the deviation amplitude (or radius), defined, respectively, as:

$$\alpha_{0,i} = \frac{1}{2}(\underline{\alpha}_i + \overline{\alpha}_i); \quad \Delta\alpha_i = \frac{1}{2}(\overline{\alpha}_i - \underline{\alpha}_i). \tag{5a,b}$$

However, in structural engineering, the uncertain-but-bounded parameters can be reasonably assumed to be symmetric, i.e. $\overline{\alpha}_i = -\underline{\alpha}_i \equiv \alpha_i$, so that:

$$\alpha_{0,i} = \frac{\overline{\alpha}_i + \underline{\alpha}_i}{2} = 0; \qquad \Delta\alpha_i = \frac{\overline{\alpha}_i - \underline{\alpha}_i}{2} \equiv \alpha_i > 0. \tag{6a,b}$$

Under this assumption, the generic interval variable can be written in *affine form* as:

$$\alpha_i^I = \Delta\alpha_i\,\hat{e}_i^I. \tag{7}$$

Following the interval formalism above introduced, the interval stiffness matrix $\mathbf{K}(\boldsymbol{\alpha}^I)$ can be expressed as a linear function of the dimensionless interval parameters α_i^I, i.e.:

$$\mathbf{K}(\boldsymbol{\alpha}) = \mathbf{K}_0 + \Delta\mathbf{K}(\boldsymbol{\alpha}), \qquad \boldsymbol{\alpha} \in \boldsymbol{\alpha}^I = [\underline{\boldsymbol{\alpha}}, \overline{\boldsymbol{\alpha}}] \tag{8}$$

where $\mathbf{K}_0$ is the nominal stiffness matrix, which is a positive definite symmetric matrix of order $n \times n$ and $\Delta\mathbf{K}(\boldsymbol{\alpha}^I)$ is the interval deviation of the stiffness matrix with respect to the nominal one.

To solve the dynamical problem (2) involving the interval parameters (7), it is required to find at each time instant an interval vector containing the dynamical response set, i.e.

$$\mathbf{u}(\boldsymbol{\alpha}^I, t) = [\underline{\mathbf{u}}(t), \overline{\mathbf{u}}(t)] \tag{9}$$

or in component form

$$u_j(\boldsymbol{\alpha}^I, t) = [\underline{u}_j(t), \overline{u}_j(t)], \quad j = 1, 2, \ldots, n \tag{10}$$

with

$$\begin{aligned} \underline{u}_j(t) &= \min\{u_j(\boldsymbol{\alpha}, t) \,|\, u_j(\boldsymbol{\alpha}, t) \in \mathbb{R}, \boldsymbol{\alpha} \in \mathbb{IR}^r\}, \\ \overline{u}_j(t) &= \max\{u_j(\boldsymbol{\alpha}, t) \,|\, u_j(\boldsymbol{\alpha}, t) \in \mathbb{R}, \boldsymbol{\alpha} \in \mathbb{IR}^r\}; \quad j = 1, 2, \ldots, n \end{aligned} \tag{11a,b}$$

where the symbols $\min\{\bullet\}$ and $\max\{\bullet\}$ denote minimum (inferior) and maximum (superior) value, respectively, while $\{S(\boldsymbol{\alpha}) | P(\boldsymbol{\alpha})\}$ means "the set of quantities $S(\boldsymbol{\alpha})$ such that the proposition $P(\boldsymbol{\alpha})$ holds".

In the next section, a method to evaluate in approximate form the interval dynamic response is described.

3 BOUNDS OF INTERVAL DYNAMIC RESPONSES

The first step in vibration analysis of structures is the solution of an eigenproblem for evaluating the natural frequencies and the associated mode shapes. For classically damped linear discretized structures with r uncertain-but-bounded parameters a generalized interval eigenvalue problem must be solved [19-22]:

$$\mathbf{K}(\boldsymbol{\alpha})\boldsymbol{\phi}_j(\boldsymbol{\alpha}) = \lambda_j(\boldsymbol{\alpha})\mathbf{M}\,\boldsymbol{\phi}(\boldsymbol{\alpha}); \qquad \boldsymbol{\alpha} \in \boldsymbol{\alpha}^I = [\underline{\boldsymbol{\alpha}}, \overline{\boldsymbol{\alpha}}], \quad (j = 1, 2, \ldots, n) \tag{12}$$

where $\mathbf{K}(\boldsymbol{\alpha}^I)$ is the $n \times n$ stiffness matrix of the structural system which depends on the dimensionless uncertain parameters collected into the interval vector $\boldsymbol{\alpha}^I \in \mathbb{IR}^r$; $\mathbf{M}$ is the $n \times n$ mass matrix; $\lambda_j(\boldsymbol{\alpha}^I) = \omega_j^2(\boldsymbol{\alpha}^I)$ is the $j-$th interval eigenvalue, equivalent to the squared interval natural frequency, and $\boldsymbol{\phi}_j(\boldsymbol{\alpha}^I)$ is the associated interval eigenvector.

According to the *classical interval analysis* [16, 17], the interval stiffness matrix satisfies the following relationship:

$$\mathbf{K}(\boldsymbol{\alpha}^I) = [\underline{\mathbf{K}}, \overline{\mathbf{K}}] = \{\mathbf{K}(\boldsymbol{\alpha}) \,|\, \underline{k}_{ij} \le k_{ij} \le \overline{k}_{ij}\} \tag{13}$$

where $\underline{k}_{ij}$ and $\overline{k}_{ij}$ are the $(i, j)-$th element of the stiffness matrices $\underline{\mathbf{K}}$ and $\overline{\mathbf{K}}$, respectively. In vibration problems, $\mathbf{K} \in \mathbf{K}(\boldsymbol{\alpha}^I)$ is a symmetric positive definite matrix.

The solution of the generalized interval eigenvalue problem involves the evaluation of all possible eigenvalues satisfying Eq.(12) as the matrix $\mathbf{K}(\boldsymbol{\alpha}^I)$ assumes all possible values inside the intervals defined in Eq.(13). The solutions constitute a complicated region in the real number field $\mathbb{R}$. Therefore, the objective is to evaluate, for each eigensolution, the narrowest interval enclosing all possible eigenvalues satisfying Eq.(12), i.e. [19-22]:

$$\lambda_j(\boldsymbol{\alpha}) = \omega_j^2(\boldsymbol{\alpha}) = \left[\underline{\lambda}_j, \overline{\lambda}_j\right], \qquad \boldsymbol{\alpha} \in \boldsymbol{\alpha}^I = \left[\underline{\boldsymbol{\alpha}}, \overline{\boldsymbol{\alpha}}\right] \tag{14}$$

where $\underline{\lambda}_j$ and $\overline{\lambda}_j$, $(j = 1, 2, \ldots, n)$, are the LB and UB of the $j-$th interval eigenvalue.

The eigenvectors associated with the interval eigenvalues are also affected by the uncertainties and turn out to be bounded by interval vectors $\boldsymbol{\phi}_j \in \boldsymbol{\phi}_j(\boldsymbol{\alpha}^I)$.

Since the eigenvalues are monotonic functions of the uncertain parameters $\alpha_j \in \alpha_j^I = \left[\underline{\alpha}_j, \overline{\alpha}_j\right]$, $(j = 1, 2, \ldots, r)$, then the bounds of the eigenvalues can be evaluated solving the following two deterministic eigenvalue problems [22]:

$$\begin{aligned} &\mathbf{K}\left(\underline{\boldsymbol{\alpha}}\right)\boldsymbol{\phi}_j^{(\mathrm{LB})} = \underline{\lambda}_j\,\mathbf{M}\,\boldsymbol{\phi}_j^{(\mathrm{LB})}\,; \quad \boldsymbol{\phi}_j^{(\mathrm{LB})T}\,\mathbf{M}\,\boldsymbol{\phi}_k^{(\mathrm{LB})} = \Delta_{jk} \\ &\mathbf{K}\left(\overline{\boldsymbol{\alpha}}\right)\boldsymbol{\phi}_j^{(\mathrm{UB})} = \overline{\lambda}_j\,\mathbf{M}\,\boldsymbol{\phi}_j^{(\mathrm{UB})}; \quad \boldsymbol{\phi}_j^{(\mathrm{UB})T}\,\mathbf{M}\,\boldsymbol{\phi}_k^{(\mathrm{LB})} = \Delta_{jk}\,, \quad (j = 1, 2, \ldots n). \end{aligned} \tag{15a,b}$$

where Δ_{jk} is the Kronecker delta; $\boldsymbol{\phi}_j^{(\mathrm{LB})}$ and $\boldsymbol{\phi}_j^{(\mathrm{UB})}$ are the eigenvectors associated to the eigenproblem in which $\boldsymbol{\alpha} = \underline{\boldsymbol{\alpha}}$ and $\boldsymbol{\alpha} = \overline{\boldsymbol{\alpha}}$, respectively. Notice that the two stiffness matrices $\mathbf{K}\left(\underline{\boldsymbol{\alpha}}\right)$ and $\mathbf{K}\left(\overline{\boldsymbol{\alpha}}\right)$ as well as the mass matrix $\mathbf{M}$ are real, symmetric and positive definite matrices. Then, the eigenvectors of both eigenproblems are real vectors while the eigenvalues are real and positive quantities.

Introducing the diagonal matrices $\underline{\boldsymbol{\Omega}}^2$ and $\overline{\boldsymbol{\Omega}}^2$ whose $j-$th element is $\underline{\lambda}_j$ and $\overline{\lambda}_j$, respectively, and the matrices $\boldsymbol{\Phi}^{(\mathrm{LB})}$ and $\boldsymbol{\Phi}^{(\mathrm{UB})}$ whose $j-$th column is $\boldsymbol{\phi}_j^{(\mathrm{LB})}$ and $\boldsymbol{\phi}_j^{(\mathrm{UB})}$, respectively, it is possible to evaluate the response of quiescent structural systems in integral form as follows [23, 24]:

$$\begin{aligned} \mathbf{y}\left(\underline{\boldsymbol{\alpha}}, t\right) &= \int_0^t \boldsymbol{\Theta}\left(\underline{\boldsymbol{\alpha}}, t-\tau\right)\mathbf{V}\,\mathbf{f}(\tau)\,\mathrm{d}\tau\,; \\ \mathbf{y}\left(\overline{\boldsymbol{\alpha}}, t\right) &= \int_0^t \boldsymbol{\Theta}\left(\overline{\boldsymbol{\alpha}}, t-\tau\right)\mathbf{V}\,\mathbf{f}(\tau)\,\mathrm{d}\tau \end{aligned} \tag{16a,b}$$

where $\mathbf{y}\left(\underline{\boldsymbol{\alpha}}, t\right)$ and $\mathbf{y}\left(\overline{\boldsymbol{\alpha}}, t\right)$ are the state variable vectors defined as:

$$\mathbf{y}\left(\underline{\boldsymbol{\alpha}}, t\right) = \begin{bmatrix} \mathbf{u}\left(\underline{\boldsymbol{\alpha}}, t\right) \\ \dot{\mathbf{u}}\left(\underline{\boldsymbol{\alpha}}, t\right) \end{bmatrix}; \quad \mathbf{y}\left(\overline{\boldsymbol{\alpha}}, t\right) = \begin{bmatrix} \mathbf{u}\left(\overline{\boldsymbol{\alpha}}, t\right) \\ \dot{\mathbf{u}}\left(\overline{\boldsymbol{\alpha}}, t\right) \end{bmatrix} \tag{17a,b}$$

and $\mathbf{V}$ is the following $2n \times n$ matrix:

$$\mathbf{V} = \begin{bmatrix} \mathbf{0} \\ \mathbf{M}^{-1} \end{bmatrix}. \tag{18}$$

In Eqs.(16), $\boldsymbol{\Theta}\left(\underline{\boldsymbol{\alpha}}, t\right)$ and $\boldsymbol{\Theta}\left(\overline{\boldsymbol{\alpha}}, t\right)$ are the $2n \times 2n$ transition matrices given, respectively, by:

$$\boldsymbol{\Theta}\left(\underline{\boldsymbol{\alpha}},t\right)=\begin{bmatrix}-\boldsymbol{\Phi}^{(\mathrm{LB})}\mathbf{g}\left(\underline{\boldsymbol{\alpha}},t\right)\boldsymbol{\Phi}^{(\mathrm{LB})T}\mathbf{K}\left(\underline{\boldsymbol{\alpha}}\right) & \boldsymbol{\Phi}^{(\mathrm{LB})}\dot{\mathbf{g}}\left(\underline{\boldsymbol{\alpha}},t\right)\boldsymbol{\Phi}^{(\mathrm{LB})T}\mathbf{M}\\ -\boldsymbol{\Phi}^{(\mathrm{LB})}\dot{\mathbf{g}}\left(\underline{\boldsymbol{\alpha}},t\right)\boldsymbol{\Phi}^{(\mathrm{LB})T}\mathbf{K}\left(\underline{\boldsymbol{\alpha}}\right) & \boldsymbol{\Phi}^{(\mathrm{LB})}\ddot{\mathbf{g}}\left(\underline{\boldsymbol{\alpha}},t\right)\boldsymbol{\Phi}^{(\mathrm{LB})T}\mathbf{M}\end{bmatrix};$$

$$\boldsymbol{\Theta}\left(\overline{\boldsymbol{\alpha}},t\right)=\begin{bmatrix}-\boldsymbol{\Phi}^{(\mathrm{UB})}\mathbf{g}\left(\overline{\boldsymbol{\alpha}},t\right)\boldsymbol{\Phi}^{(\mathrm{UB})T}\mathbf{K}\left(\overline{\boldsymbol{\alpha}}\right) & \boldsymbol{\Phi}^{(\mathrm{UB})}\dot{\mathbf{g}}\left(\overline{\boldsymbol{\alpha}},t\right)\boldsymbol{\Phi}^{(\mathrm{UB})T}\mathbf{M}\\ -\boldsymbol{\Phi}^{(\mathrm{UB})}\dot{\mathbf{g}}\left(\overline{\boldsymbol{\alpha}},t\right)\boldsymbol{\Phi}^{(\mathrm{UB})T}\mathbf{K}\left(\overline{\boldsymbol{\alpha}}\right) & \boldsymbol{\Phi}^{(\mathrm{UB})}\ddot{\mathbf{g}}\left(\overline{\boldsymbol{\alpha}},t\right)\boldsymbol{\Phi}^{(\mathrm{UB})T}\mathbf{M}\end{bmatrix}. \qquad (19a,b)$$

In the previous equations, $\mathbf{g}\left(\underline{\boldsymbol{\alpha}},t\right)$ and $\mathbf{g}\left(\overline{\boldsymbol{\alpha}},t\right)$ are two diagonal matrices whose j-th element can be evaluated, respectively, as:

$$g_j\left(\underline{\boldsymbol{\alpha}},t\right)=-\frac{1}{\underline{\omega}_j^2}\exp(-\underline{\xi}_j\underline{\omega}_j t)\left[\cos\left(\underline{\omega}_j t\sqrt{1-\underline{\xi}_j^2}\right)+\frac{\underline{\xi}_j\,\underline{\omega}_j}{\underline{\omega}_j\sqrt{1-\underline{\xi}_j^2}}\mathrm{sen}\left(\underline{\omega}_j t\sqrt{1-\underline{\xi}_j^2}\right)\right];$$

$$g_j\left(\overline{\boldsymbol{\alpha}},t\right)=-\frac{1}{\overline{\omega}_j^2}\exp(-\overline{\xi}_j\overline{\omega}_j t)\left[\cos\left(\overline{\omega}_j t\sqrt{1-\overline{\xi}_j^2}\right)+\frac{\overline{\xi}_j\,\overline{\omega}_j}{\overline{\omega}_j\sqrt{1-\overline{\xi}_j^2}}\mathrm{sen}\left(\overline{\omega}_j t\sqrt{1-\overline{\xi}_j^2}\right)\right] \qquad (20a,b)$$

where $\underline{\omega}_j$ and $\overline{\omega}_j$ denote the $j-$th element of the diagonal matrices $\underline{\boldsymbol{\Omega}}$ and $\overline{\boldsymbol{\Omega}}$, respectively; $\underline{\xi}_j=(c_M+c_K\,\underline{\omega}_j^{\,2})/2\underline{\omega}_j$ and $\overline{\xi}_j=(c_M+c_K\,\overline{\omega}_j^{\,2})/2\overline{\omega}_j$ are the LB and UB of the $j-$th damping ratio under the Rayleigh condition (3).

Finally, the LB, $\underline{\mathbf{y}}(t)$, and UB, $\overline{\mathbf{y}}(t)$, of the state variable dynamic response vectors can be obtained as:

$$\underline{\mathbf{y}}(t)=\min\left\{\mathbf{y}\left(\underline{\boldsymbol{\alpha}},t\right),\,\mathbf{y}\left(\overline{\boldsymbol{\alpha}},t\right)\right\};$$
$$\overline{\mathbf{y}}(t)=\max\left\{\mathbf{y}\left(\underline{\boldsymbol{\alpha}},t\right),\,\mathbf{y}\left(\overline{\boldsymbol{\alpha}},t\right)\right\}. \qquad (21a,b)$$

In the previous equations, the symbols $\min\{\bullet\}$ and $\max\{\bullet\}$ mean minimum (inferior) and maximum (superior) value component wise, respectively.

It is worth mentioning that the exact bounds of the dynamic response are not generally obtained setting all the uncertain parameters simultaneously to their LB and UB. Hence, more accurate results may be achieved by introducing in the present formulation the most common combinations of the extreme values of the interval variables α_i^I among those detected at each time instant by performing a preliminary sensitivity analysis.

4 NUMERICAL APPLICATIONS

The first application concerns the 3D 26-bar truss structure with 18 DOFs, subjected to an impulsive load $f(t)=1000\,\delta(t)$ N, as shown in Figure 1. The following geometrical and mechanical properties are assumed: nominal cross-sectional area of the bars $A_0=A_{0,i}=4.27\times10^{-4}$ m^2 and nominal Young's moduli $E_0=E_{0,i}=2.1\times10^{8}$ kN/m^2, $i=1,2,\ldots,26$. Young's moduli of $r=13$ bars are modeled as interval variables, $E_i^I=E_0(1+\Delta\alpha\,\hat{e}_i^I)$, $i=1,2,\ldots,13$ (see bar numbering in Figure 1).

According to Eq. (16), the two dynamical responses under the impulsive load, pertaining to the LB and UB of the uncertain parameters, can be evaluated as:

$$\mathbf{y}\left(\underline{\boldsymbol{\alpha}},t\right)=\boldsymbol{\Theta}\left(\underline{\boldsymbol{\alpha}},t\right)\mathbf{v};\quad \mathbf{y}\left(\overline{\boldsymbol{\alpha}},t\right)=\boldsymbol{\Theta}\left(\overline{\boldsymbol{\alpha}},t\right)\mathbf{v} \qquad (22a,b)$$

where $\mathbf{v}$ is a 18×1 vector having all elements equal to zero except the 18-th one which is equal to $1000\ \mathrm{N}$.

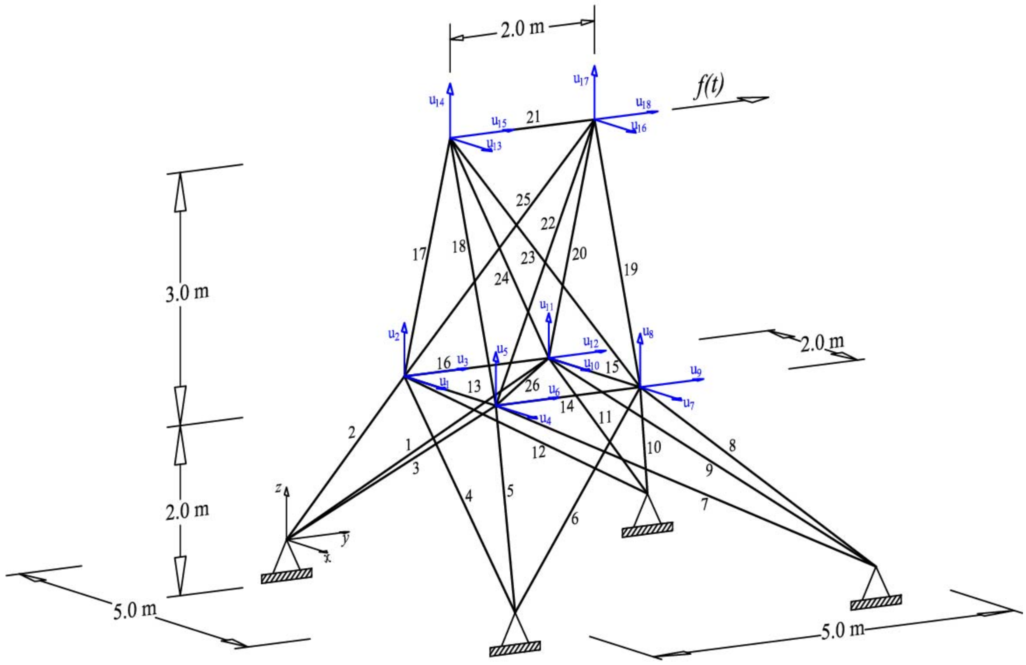

Figure 1. 3D 26-bar truss structure with uncertain Young's moduli of $r = 13$ bars.

In Figures 2 and 3, the LB and UB time-histories of the horizontal displacement $u_{18}(t)$ are plotted for two deviation amplitudes of the interval uncertainties, say $\Delta\alpha = 0.1$ and $\Delta\alpha = 0.2$, respectively. The proposed bounds are compared with the ones obtained by applying the *IPM* as well as with the exact bounds provided by the Vertex Method. The latter requires to perform the deterministic analysis in the time domain for all possible combinations of the bounds of the interval Young's moduli, say 2^r, and then take at each time instant the minimum and maximum among the corresponding responses. By inspection of these figures, it can be observed that the proposed method is able to predict the bounds of the dynamic responses with great accuracy. Furthermore, it is worth remarking that the proposed method is much more accurate than the *IPM* as the degree of uncertainty increases.

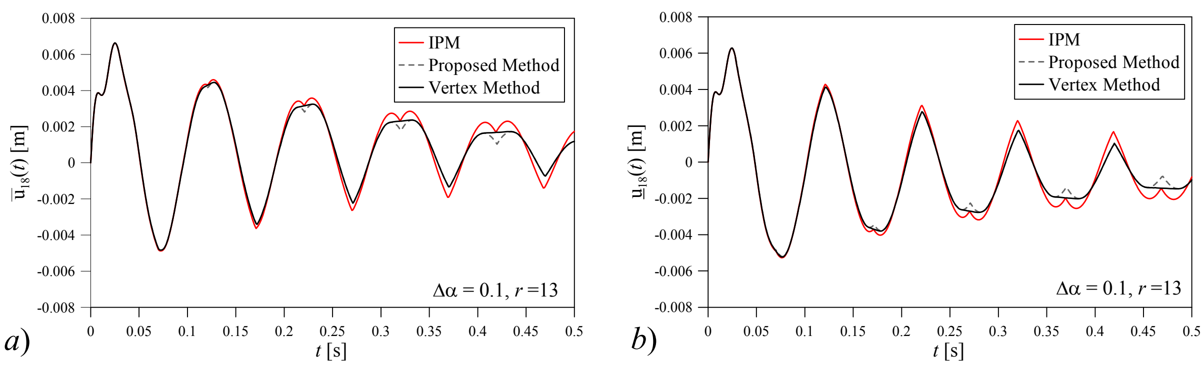

Figure 2. Time-histories of the a) UB and b) LB of the horizontal displacement $u_{18}(t)$ for a deviation amplitude of the uncertain parameters $\Delta\alpha = 0.1$: comparison between *IPM*, Proposed method and Vertex Method.

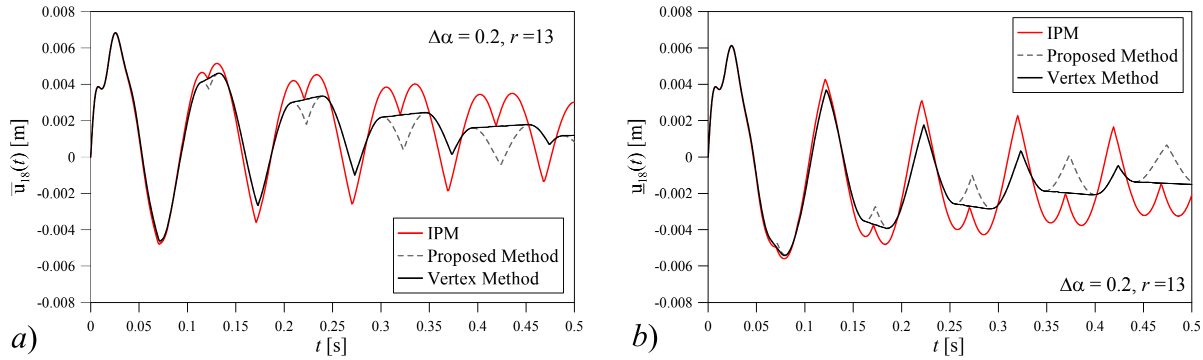

Figure 3. Time-histories of the a) UB and b) LB of the horizontal displacement $u_{18}(t)$ for a deviation amplitude of the uncertain parameters $\Delta\alpha = 0.2$: comparison between *IPM*, Proposed method and Vertex Method.

As second application the shear-type frame depicted in Figure 4 is analyzed.

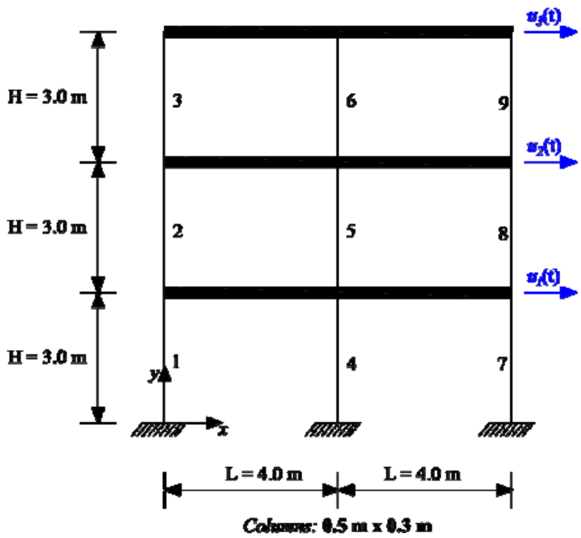

Figure 4. Geometric configuration of three-story two-bays shear-type frame.

This frame has a uniform story height $H = 3.0$ m and a bay width $L = 6.0$ m, as shown in Figure 4. The beams are considered rigid to enforce a typical shear building behaviour. Under this assumptions, the shear-frame is modelled as a three DOFs linear system. The tributary mass per story, *M*, accounting for the structure's own weight, as well as for permanent and live loads, is equal to $M = 81\,520$ kg . Young's moduli of columns are modelled as uncertain-but-bounded parameters, $E_i^I = E_0(1+\Delta\alpha\, \hat{e}_i^I)$, $i = 1,2,\ldots,9$, with $\Delta\alpha$ denoting the deviation amplitude and $E_0 = E_{0,i} = 3.15\times 10^7$ kN/m^2 . The frame is subjected to the seismic acceleration recorded at El Centro (1940).

The accuracy of the presented method can be detected by inspection of Figures 5 and 6, where the proposed time-histories of the UB and LB of the displacements of the third floor, $\underline{u}_3(t)$ and $\overline{u}_3(t)$, are compared with the exact and *IPM* solutions for two different deviation amplitudes of the uncertain parameters, say $\Delta\alpha = 0.1$ and $\Delta\alpha = 0.2$. The exact bounds are evaluated following the philosophy of the Vertex Method. Notice that, contrary to the *IPM*, the proposed method gives accurate estimates of the LB and UB of the response even when relatively large uncertainties are involved. It is worth mentioning that the presented procedure is much more efficient from a computational point of view than the Vertex Method, especially when a large number of uncertain parameters is considered.

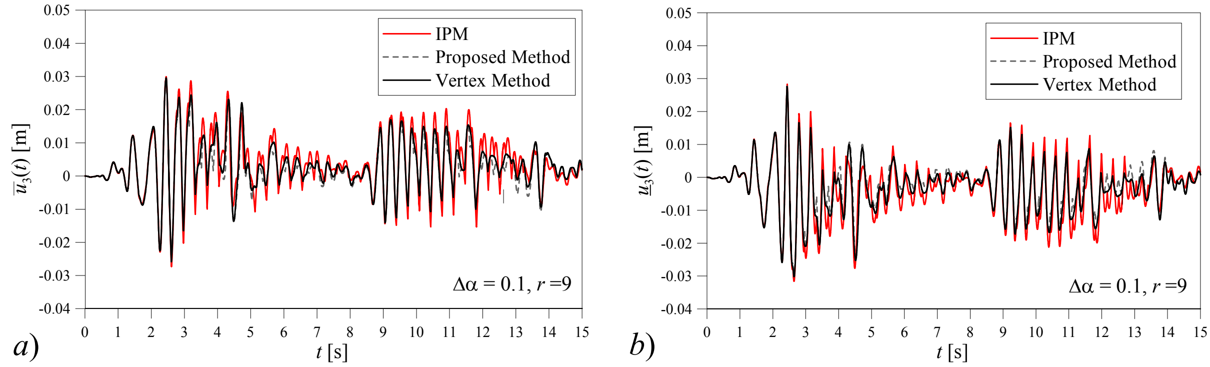

Figure 5. Time-histories of the a) UB and b) LB of the horizontal displacement $u_3(t)$ for a deviation amplitude of the uncertain parameters $\Delta\alpha = 0.1$: comparison between *IPM*, Proposed method and Vertex Method.

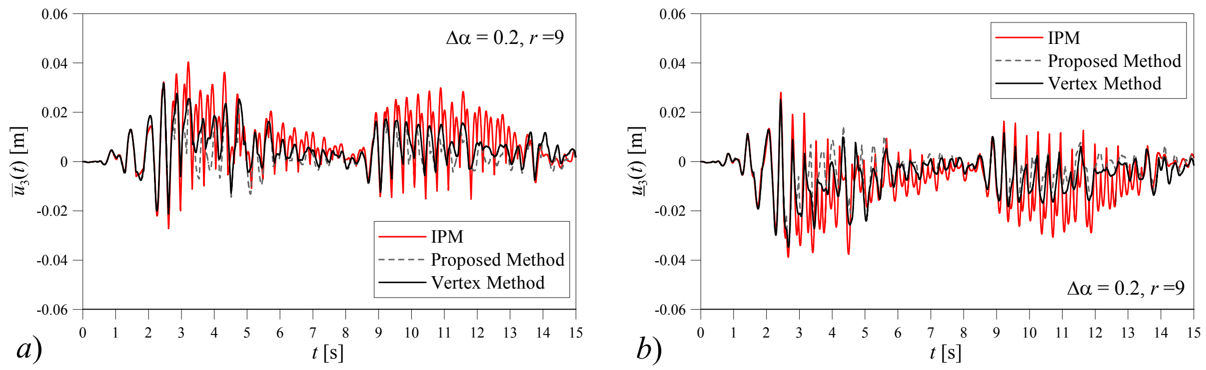

Figure 6. Time-histories of the a) UB and b) LB of the horizontal displacement $u_3(t)$ for a deviation amplitude of the uncertain parameters $\Delta\alpha = 0.2$: comparison between *IPM*, Proposed method and Vertex Method.

5 CONCLUSIONS

In this paper, a novel procedure for evaluating the bounds of the response of linear structural systems with uncertain-but-bounded properties, subjected to dynamic deterministic excitations has been proposed. The proposed procedure is able to overcome the main limitations of the *IPM*, as shown by numerical results.

The main steps required by the proposed approach are: i) to solve two deterministic eigenvalue problems, where the uncertainties are set to the LB, $\underline{\boldsymbol{\alpha}}$, and UB, $\overline{\boldsymbol{\alpha}}$, in order to obtain the LB and UB of the eigenvalues; ii) to compute two transition matrices; iii) to compute the two state variable responses $\mathbf{y}(\underline{\boldsymbol{\alpha}},t)$ and $\mathbf{y}(\overline{\boldsymbol{\alpha}},t)$; iv) to evaluate the LB and UB of the structural response by handy formulas.

Numerical results have demonstrated that the proposed method provides estimates of the bounds of the dynamic response very close to the exact ones, even for relatively large uncertainty levels.

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