Available online at www.eccomasproceedia.org
Eccomas Proceedia UNCECOMP (2017) 329-338

ECCOMAS
Proceedia

UNCECOMP 2017
2nd ECCOMAS Thematic Conference on
Uncertainty Quantification in Computational Sciences and Engineering
M. Papadrakakis, V. Papadopoulos, G. Stefanou (eds.)
Rhodes Island, Greece, 15–17 June 2017

REAL-TIME FUZZY ANALYSIS OF MACHINE DRIVEN TUNNELING

Ba Trung Cao[1], **Steffen Freitag**[1], **and Günther Meschke**[1]

[1]Institute for Structural Mechanics, Ruhr University Bochum
Universitätstrasse 150, 44801, Bochum, Germany
e-mail: ba.cao@rub.de; steffen.freitag@sd.rub.de and guenther.meschke@rub.de

Keywords: Surrogate Model, Mechanized Tunneling, Uncertainty Quantification, Fuzzy Analysis.

Abstract. *Reliability assessment in mechanized tunneling requires to take into account limited information describing the local geology and the corresponding geotechnical parameters. The geotechnical data are often quite limited and generally not available in the form of precise models and parameter values. In this case, epistemic uncertainty should be considered within the reliability assessment. The concept of fuzzy numbers is applied to predict tunneling induced settlements in real-time. An advanced numerical simulation is utilized as forward model for the settlement prediction. However, to achieve real-time capabilities, surrogate models are required. Deterministic surrogate models can be used together with an* α*-cut optimization approach to compute fuzzy data. In this paper, a surrogate modeling strategy is introduced to directly process fuzzy input-output data. Within this approach, the time-consuming optimization procedure is replaced by a surrogate model to obtain the fuzzy settlement field prediction in real-time. The significant reduction in computation time maintaining similar prediction performance leads to potential applications in steering of mechanized tunneling processes.*

Peer-review under responsibility of the organizing committee of UNCECOMP 2017.
doi: 10.7712/120217.5373.17151

1 INTRODUCTION

In tunnel projects, the information of the local geology and the associated soil conditions are often limited and imprecise. Therefore, a reliability assessment procedure is required, which also considers epistemic source of uncertainty, i.e. by means of intervals, fuzzy numbers, and imprecise probability approaches. General concepts for reliability analyses in mechanized tunneling are presented e.g. in [1].

To perform these analyses a forward model is required, which is able to capture the system response and the various interactions between individual components during the tunneling process. Currently, simulations based on the Finite Element (FE) method are used to investigate and predict the soil-structure interactions in mechanized tunneling. In this paper, a three dimensional FE model for simulations of shield-driven tunnels in soft soil, see [2], is applied. Taking into account fuzzy numbers for the geotechnical parameters, multiple runs of the simulation model are required within the α-cut optimization. The FE simulation must be replaced by surrogate models to reduce the computation time significantly, especially for real-time applications while maintaining the prediction performance of the original FE simulation model.

In [3], a hybrid surrogate model based on a combination of Recurrent Neural Networks (RNN) and Proper Orthogonal Decomposition (POD) is proposed for deterministic input-output mapping with high-dimensional outputs. The hybrid surrogate model has been utilized together with Particle Swarm Optimization to perform optimization based interval analyses for computing the time variant interval settlement field with more than 100 surface points. To obtained results with an optimization based interval analysis takes several hours, which is not practical for real-time applications in mechanized tunneling. To enable real-time capabilities, an extension of the hybrid surrogate model is presented in [4] to directly map interval input-output data, which are expressed by a midpoint-radius representation. The key idea of the strategy is to also replace the complete time-consuming optimization loop by surrogate models and not only the deterministic FE simulation model. The results of an application example show a significant reduction in computation time with similar accuracy as the optimization based approach. In this paper, this approach is further extended for time variant fuzzy settlement field predictions in real-time. Fuzzy numbers are expressed by α-cuts and a Δ-representation of the bounds, which requires use the Non-Negative Matrix Factorization (NNMF) technique instead of the POD within some sub-surrogate models. The performance of the proposed strategy is illustrated through a fuzzy analyses and also within a fuzzy probability box analyses of a mechanized tunneling process.

2 UNCERTAINTY QUANTIFICATION

2.1 Intervals and fuzzy numbers

The epistemic source of uncertainty can be quantified by intervals or fuzzy numbers. An interval

$$\bar{x} = \left[{}^{l}x, {}^{u}x\right] \tag{1}$$

is defined by its lower bound ${}^{l}x$ and upper bound ${}^{u}x$ without further assumptions of a distribution within this possible range. By defining a membership function $\mu(x)$ of the uncertain set, the concept of intervals is extended to fuzzy numbers. If the membership function is discretised by α-cuts, a set of nested intervals is obtained, see Fig. 1. In general, computations with fuzzy numbers can be performed similar to interval analyses, i.e. by means of fuzzy arithmetic [5] or optimization based approaches [6], see also [7] for an overview.

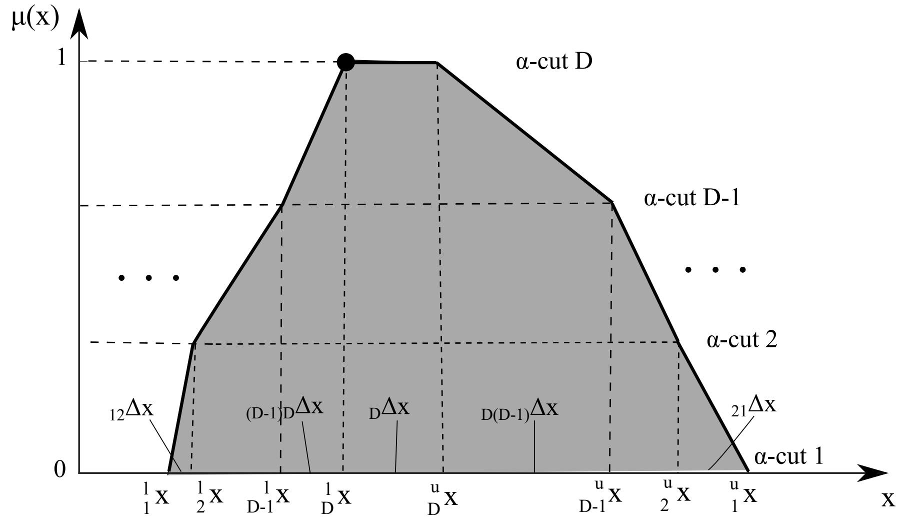

Figure 1: A typical fuzzy number with D α-cuts.

A typical fuzzy number with D α-cuts, as shown in Fig. 1, can be represented by a sorted sequence containing all lower and upper bounds

$$\tilde{x} = \langle {}^l_1 x, ..., {}^l_D x, {}^u_D x, ..., {}^u_1 x \rangle. \tag{2}$$

The fuzzy number can also be expressed by defining a reference point, e.g. ${}^l_D x$, and incremental differences Δ to all other bounds, see e.g. [8].

$$\tilde{x} = \langle {}^l_D x, {}_{12}\Delta x, ..., {}_{(D-1)D}\Delta x, {}_D\Delta x, {}_{D(D-1)}\Delta x, ..., {}_{21}\Delta x \rangle. \tag{3}$$

Whereas the reference point ${}^l_D x$ is defined in $\mathbb{R}$, all Δ values in the above equation must be positive numbers. In this paper, the fuzzy settlement field, i.e. the vector $\widetilde{\mathbf{S}}$ containing all settlements of a surface area, will be described by the Δ-representation as

$$\widetilde{\mathbf{S}} = \langle {}^l_D \mathbf{S}, {}_{12}\Delta \mathbf{S}, ..., {}_{(D-1)D}\Delta \mathbf{S}, {}_D\Delta \mathbf{S}, {}_{D(D-1)}\Delta \mathbf{S}, ..., {}_{21}\Delta \mathbf{S} \rangle. \tag{4}$$

The type of input-output mapping used in this paper

$$\mathbf{P}(t) \mathrel{\widetilde{\mapsto}} \widetilde{\mathbf{S}}(t) \tag{5}$$

means that the time variant deterministic steering parameters $\mathbf{P}(t)$ are defined as inputs and mapped with fuzzy model parameters onto the time variant fuzzy settlement field $\widetilde{\mathbf{S}}(t)$. The fuzzy model parameters of the mapping model contain the influence of the time constant geotechnical fuzzy parameters.

2.2 Probability box approach

In case of limited statistical information, e.g. estimation of stochastic models with small sample sizes, imprecise probability concepts can be adopted to quantify the variability of uncertain parameters. The probability box (p-box) approach, see e.g. [9], can be used to define imprecise stochastic numbers by its lower bound ${}^l F(x)$ and upper bound ${}^u F(x)$ cumulative distribution function (cdf). In general, arbitrary stochastic models, including empirical distributions, can be used for the lower and upper bound cdf.

Here, a fuzzy stochastic approach for real-time predictions of time variant settlements in mechanized tunneling is developed. This is realized by running a stochastic analysis (i.e. Monte Carlo simulations) with fuzzy samples. At each α-cut, the lower bound ${}^{l}F(x)$ and upper bound ${}^{u}F(x)$ cdfs of the structural responses are computed. As can be seen in the left part of Fig. 2, for each Monte Carlo run, a deterministic tunneling simulation model has to be computed taking fuzzy soil parameters into account, i.e. for each settlement component, two optimization problems have to be solved at each α-cut. Due to the high computational effort, a deterministic surrogate model is created to replace the FE simulation model, see middle part of Fig. 2. In order to achieve real time performance, also the optimization loops are replaced by a fuzzy surrogate model, see right part of Fig. 2. This allows to significantly reduce the computation time within the p-box approach.

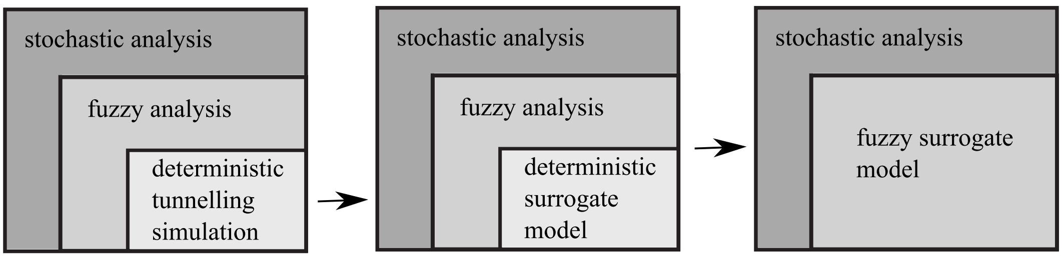

Figure 2: Reliability analyses with polymorphic uncertain data (p-box approach).

3 HYBRID SURROGATE MODEL STRATEGY

The new surrogate modeling scheme for real-time predictions with fuzzy numbers in mechanized tunneling is depicted in Fig. 3. In the offline stage, a surrogate model of a tunnel section is generated using FE simulation results based on deterministic inputs (realizations $\mathbf{X}$ of the fuzzy geotechnical parameters $\widetilde{\mathbf{X}}$ and time variant steering parameters $\mathbf{P}(t)$) and deterministic outputs (time variant surface settlement field $\mathbf{S}(t)$). By varying the input parameters of the numerical model, a set of deterministic input-output data is collected. Afterwards, the time variant fuzzy settlement fields corresponding to a fixed fuzzy number of $\widetilde{\mathbf{X}}$ and different scenarios of $\mathbf{P}(t)$ are computed within fuzzy analyses using an optimization approach, e.g. particle swarm optimization. For real-time application, the obtained results are used to create a hybrid RNN-GPOD surrogate model for the mapping according to Eq. (5) based on the Δ-representation, see Eq. (4).

In the online stage, the surrogate model is operated to predict the fuzzy bounds of the expected surface settlement field of the next time step $n+1$. Trained RNNs are employed to predict the fuzzy bounds of settlements at several monitoring points for the next time step $n+1$. The complete time variant fuzzy surface settlement field from time step 1 to n is approximated by trained POD-Radial Basis Functions (POD-RBF) for ${}^{l}_{D}\mathbf{S}$ and NNMF-RBF surrogate models for the non-negative $\Delta\mathbf{S}$ values, respectively. Finally, the GPOD and the NNMF approaches are adopted to reproduce and predict the complete fuzzy settlement field based on a combination of the results from the two previous methods. The predicted results are then included into the available fuzzy data set and the procedure is repeated for the subsequent time steps.

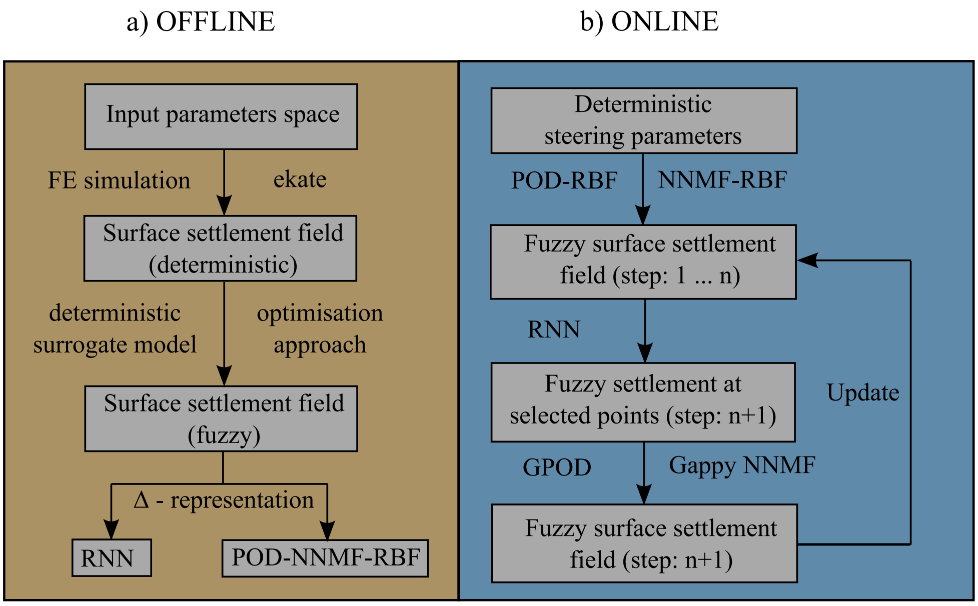

Figure 3: Scheme of the proposed hybrid surrogate modeling approach for fuzzy data.

3.1 Recurrent neural networks for fuzzy data

Fuzzy data can directly be processed with RNNs by fuzzy arithmetic operations, see e.g. [10]. Here, the Δ-representation for fuzzy data is used. For the prediction of the time variant fuzzy settlements at selected monitoring points, two types of RNNs for predicting ${}^{l}_{D}\mathbf{S}$ and the $\Delta\mathbf{S}$ values are generated. For the training procedure, the deterministic steering parameters ${}^{[n]}P$ are used as inputs in all RNNs but the target data types are different, i.e. real numbers for ${}^{l}_{D}\mathbf{S}$ and positive real numbers for all $\Delta\mathbf{S}$ values. This is realized by different activation functions in the neurons. For the RNN to predict the lower bound of α-cut D (${}^{l}_{D}\mathbf{S}$), the hyperbolic tangent function and the linear activation function are used in the hidden and output neurons, respectively. For the $\Delta\mathbf{S}$ RNNs, the logistic sigmoid function or the positive linear function is used in the output neurons to satisfy the constraint of non-negative outputs. The Levenberg-Marquardt back-propagation algorithm is employed to train the RNNs.

3.2 POD-RBF and NNMF-RBF networks for fuzzy data

The used POD approach, see [11] for an overview, and how to combine the method with RBFs to form a surrogate model are described in detail in [12]. The POD-RBF approach is used to predict ${}^{l}_{D}\mathbf{S}$ from time step 1 to n. To ensure the non-negativity constraint of the prediction of the $\Delta\mathbf{S}$ values, the POD approach is replaced by the NNMF technique.

Basically, a high-dimensional matrix $\mathbf{S}$ and a single column of $\mathbf{S}$ can be approximated as a linear combination of the truncated basis vectors $\hat{\mathbf{\Phi}}$ as $\mathbf{S} \approx \hat{\mathbf{\Phi}} \cdot \hat{\mathbf{A}}$ and $\mathbf{S}_i \approx \hat{\mathbf{\Phi}} \cdot \hat{\mathbf{A}}_i$. The truncated basis vectors are obtained from solving an eigenvalue problem of the covariance matrix $\mathbf{C} = \mathbf{S}^{\mathrm{T}} \cdot \mathbf{S}$. At this step, the truncated coefficient matrix $\hat{\mathbf{A}}$ contains constant values associated with the given matrix $\mathbf{S}$. Hence, it is only an approximation of the snapshots generated in the original high-dimensional snapshots matrix $\mathbf{S}$.

To obtain a rather continuous approximation, each coefficient vector is expressed as a non-linear function of input parameters on which the system depends. The coefficient matrix $\hat{\mathbf{A}}$ can be related to the interpolation functions by an unknown matrix of constant coefficients $\mathbf{B}$ as $\hat{\mathbf{A}} = \mathbf{B} \cdot \mathbf{F}$. In this equation, $\mathbf{F}$ contains a set of predefined interpolation functions $f_j(\mathbf{z})$ of input parameters $\mathbf{z}$. The choice of $f_j(\mathbf{z})$ can be arbitrary and in this study an inverse multiquadric radial function, a type of RBF (see [13] for a description), is chosen as interpolation function. Finally, an approximation of the output system response corresponding to an arbitrary set of input parameters is obtained by $\mathbf{S}^a \approx \hat{\mathbf{\Phi}} \cdot \mathbf{B} \cdot \mathbf{F}^a$.

The NNMF, see [14], is utilized to ensure the positive sign of the predicted results for the $\Delta\mathbf{S}^a$ values. Given a non-negative matrix $\Delta\mathbf{S}$,the NNMF algorithm is searching for two non-negative matrices $\mathbf{W}$ and $\mathbf{A}^+$ satisfying the following optimisation problem $min.\frac{1}{2} \parallel \Delta\mathbf{S} - \mathbf{W} \cdot \mathbf{A}^+ \parallel_F^2$ subject to, $\mathbf{W}, \mathbf{A}^+ \geq 0$. Similar to the POD approach, $\mathbf{W}$ and $\mathbf{A}^+$ are denoted as the *basis matrix* and *coefficient matrix*, respectively. The alternating non-negative least squares algorithm proposed in [15], which ensures the convergence of the minimization problem, is implemented in this paper. The prediction for non-negative outputs $\Delta\mathbf{S}^a$ is computed similar to the POD-RBF method as $\Delta\mathbf{S}^a \approx \mathbf{W} \cdot \mathbf{B}^+ \cdot \mathbf{F}^a$. The coefficient matrix $\mathbf{B}^+$ is the non-negative solution of the following minimization problem $min. \parallel \mathbf{A}^+ - \mathbf{B}^+ \cdot \mathbf{F} \parallel$.

3.3 GPOD and Gappy NNMF for fuzzy data

The basic POD method is combined with a linear regression called GPOD, see [16], to reconstruct the complete ${}^{l}_{D}\mathbf{S}$ values of the fuzzy settlement field. More details about the explanation and implementation of the method is given in [12] and [4]. The GPOD approach is applied to reconstruct missing elements of a given vector $\mathbf{S}^*$. It employs the concept of a gappy norm based on available data since the full norm cannot be evaluated correctly due to missing elements. The intermediate repaired vector $\mathbf{S}^*_{int}$ can be expressed in terms of truncated POD basis vectors $\hat{\mathbf{\Phi}}$ as follows $\mathbf{S}^*_{int} \approx \hat{\mathbf{\Phi}} \cdot \hat{\mathbf{A}}^*$. The coefficient vector $\hat{\mathbf{A}}^*$ can be computed by minimizing the error $E = \parallel \mathbf{S}^* - \mathbf{S}^*_{int} \parallel_n^2$. A solution to this *least squares* problem is given by a linear system of equations $\mathbf{M} \cdot \hat{\mathbf{A}}^* = \mathbf{R}$ with $\mathbf{M} = (\hat{\mathbf{\Phi}}^{\mathrm{T}}, \hat{\mathbf{\Phi}})$ and $\mathbf{R} = (\hat{\mathbf{\Phi}}^{\mathrm{T}}, \mathbf{S}^*)$.

The reconstruction procedure for a non-negative vector $\Delta\mathbf{S}^+$ follows the steps of the GPOD method with some minor modifications. The corresponding objective function $E = \parallel \Delta\mathbf{S}^+ - \mathbf{W} \cdot \Delta\mathbf{A}^+ \parallel_n^2$ to be minimised contains the distances between the available incomplete data vector and the predicted vector. The non-negative *basis matrix* $\mathbf{W}$ is assumed to be known from the available non-negative data matrix $\Delta\mathbf{S}$. The coefficient vector $\Delta\mathbf{A}^+$ is obtained considering the non-negativity constraint by solving the *non-negative least squares* problem $min. \parallel \mathbf{M}^+ \cdot \Delta\mathbf{A}^+ - \mathbf{R}^+ \parallel$ with $\mathbf{M}^+ = (\mathbf{W}^{\mathrm{T}}, \mathbf{W})$ and $\mathbf{R}^+ = (\mathbf{W}^{\mathrm{T}}, \Delta\mathbf{S}^+)$.

Finally, by replacing the missing elements in $\mathbf{S}^*$ and $\Delta\mathbf{S}^+$ by those in the corresponding reconstructed vectors, the complete fuzzy settlement field of time step $n+1$ is predicted.

4 APPLICATION IN MECHANIZED TUNNELING

The proposed surrogate modeling strategy for fuzzy data is applied to predict the time variant fuzzy settlement field of a mechanized tunneling process. The results, which are computed based on the presented Δ-representation, are compared to the reference solution (optimzation approach) in terms of prediction performance and computation time.

Figure 4(a) shows the simulation model with dimensions of 144m, 220m and 67m (in x,y,z directions). The length of each excavation step is 2m, i.e the tunnel section consists of 72 steps. The tunnel diameter and the cover depth are 10.97m and 6.5m, respectively. In this study, the elastic modulus E_1 of soil layer 1 is considered as an uncertain geotechnical parameter defined as a trapezoidal fuzzy number $\widetilde{E}_1 = \langle 52, 60, 70, 75 \rangle$ MPa. The grouting pressure ${}^{[n]}GP$ and the

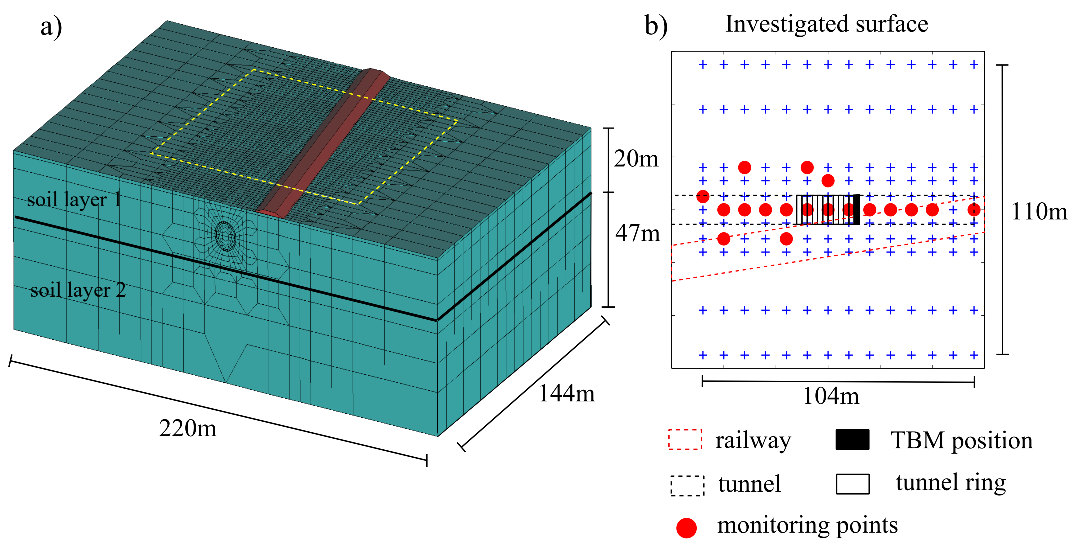

Figure 4: Numerical simulation model of a tunnel section.

support pressure ${}^{[n]}SP$ in each excavation step n are chosen as deterministic steering parameter, which can vary in each step in the corresponding ranges of 120 to 220 kPa and 100 to 200 kPa. It is assumed that the current state of the TBM advance corresponds to the 36^{th} step of the process. The fuzzy analysis performed in this paper is used to support the selection of ${}^{[n]}GP$ and ${}^{[n]}SP$ in the next time steps with the purpose to reduce tunneling induced settlements when the machine advances under an existing railway track.

An effective surface area of 110m in y-direction is investigated considering the z-displacements of 154 surface points, see Fig. 4(b), as the outputs. In the example, the fuzzy settlements of 18 selected monitoring points among these 154 surface points are predicted by the RNNs. Figure 5 shows the fuzzy settlement process of a selected surface point computed by α-cut optimization and by the proposed surrogate model for fuzzy data. The relative error of the proposed method is 3.8% in average compared to the optimization based reference solution. The most important benefit of the proposed approach is that the computation time (3 seconds) is significantly reduced compared to the optimization approach, which often required many hours.

Additionally to the fuzzy analysis, a reliability analysis adopting the p-box approach is performed with a Monte Carlo simulation using 1000 samples and the fixed $\widetilde{E}_1$. The pressures ${}^{[n]}GP$ and ${}^{[n]}SP$ are treated as stochastic processes assuming a Gaussian distribution with the mean values of 170 kN and 150 kN for ${}^{[n]}GP$ and ${}^{[n]}SP$, respectively. The standard deviations in the distribution for both pressures are considered the same at the value of σ = 30 kN. The minimum and maximum cdfs of a chosen point settlement in time step 37 corresponding to two nested intervals of $\widetilde{E}_1$ obtained from classical optimization approach and the proposed surrogate model are depicted in Fig. 6. The probability boxes obtained from the surrogate model are reasonable compared to the optimization approach. Nevertheless, the Δ-representation leads to an accumulative error for the bounds of the lower α-cuts. However, the computation time drops dramatically from around 1 day to just 20 minutes for an analysis with two α-cuts.

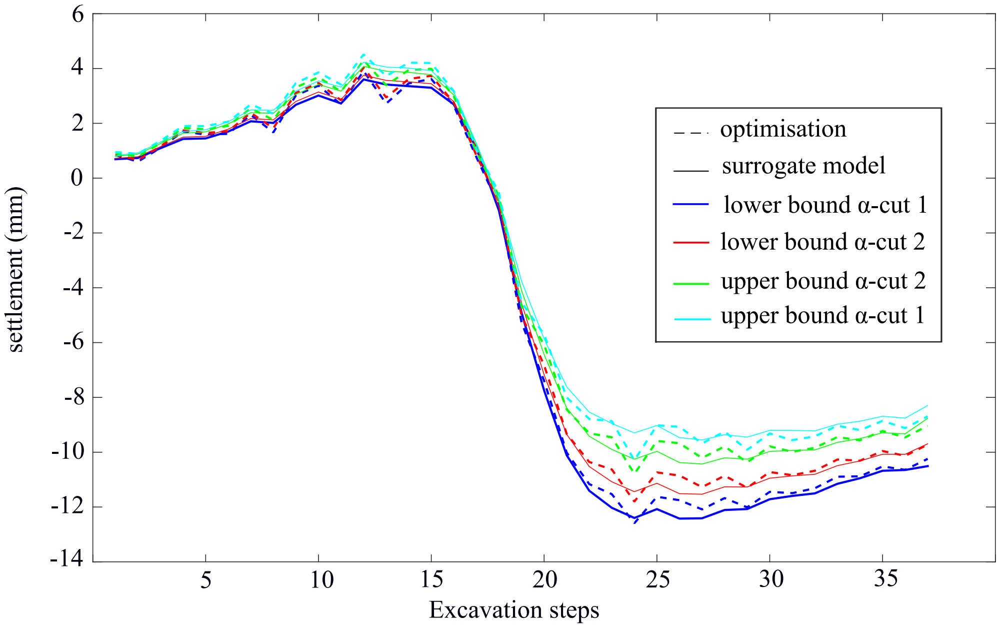

Figure 5: Fuzzy settlement in time of a selected surface point.

5 CONCLUSIONS

In this paper, the hybrid surrogate modeling for interval data [4] has been extended to process fuzzy data by means of a Δ-representation. The time-dependent behaviour of several selected points in future steps is predicted by Recurrent Neural Networks (RNNs), whereas order reduction techniques (Proper Orthogonal Decomposition and Non-Negative Matrix Factorization) are utilised to approximate the complete surface field based on the RNN predictions. The proposed method is used to predict the fuzzy surface settlements during the construction phase in mechanized tunneling for selected scenarios of the TBM steering parameters taking fuzzy geotechnical parameters into account. In comparison to the α-cut optimization approach, the computational time is significantly reduced by the proposed strategy. The new approach takes only 2 to 3 seconds to compute the fuzzy bounds of a settlement field with 154 settlement components with a similar accuracy compared to the optimization approach which requires many hours. Future developments of the proposed approach include the implementation of additional steering objectives, e.g. the tunnel face stability, controlling the tunnel lining forces and reducing the tunneling induced damage of existing infrastructure and buildings.

6 ACKNOWLEDGEMENT

Financial support was provided by the German Research Foundation (DFG) in the framework of project C1 of the Collaborative Research Center SFB 837 ”Interaction Modeling in Mechanized Tunneling”. This support is gratefully acknowledged.

REFERENCES

[1] S. Freitag, M. Beer, K.K. Phoon, J. Stascheit, J. Ninić, B.T. Cao, G. Meschke, Concepts for Reliability Analyses in Mechanised Tunnelling – Part 1: Theory. *Proceedings of the*

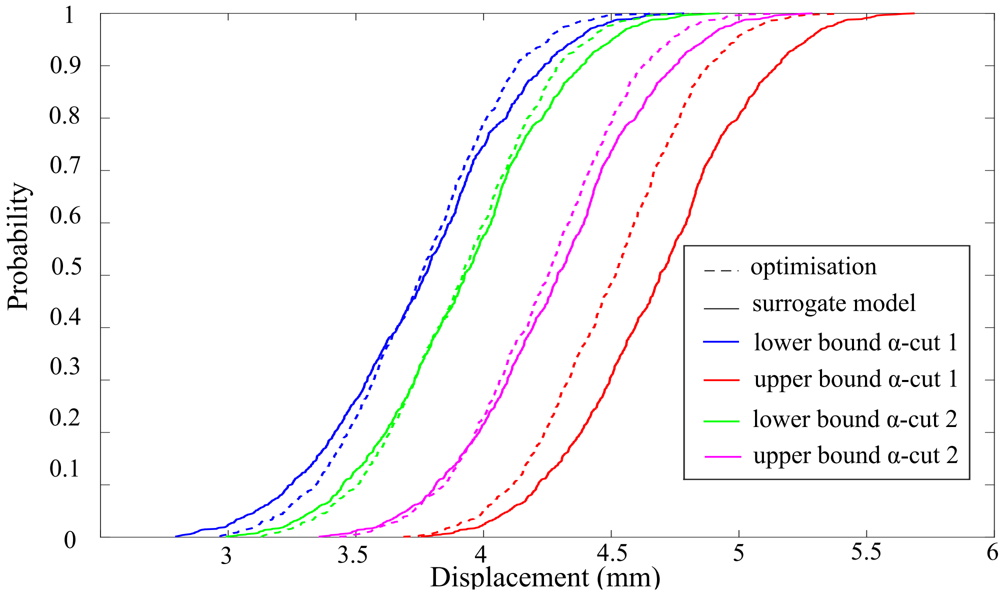

Figure 6: P-boxes for the settlement of the surface point 44 for time step 37 considering $\widetilde{E} = \langle 52, 60, 70, 75 \rangle$.

Third International Conference on Computational Methods in Tunnelling and Subsurface Engineering (EURO:TUN 2013), Bochum, Germany, 791–799, 2013.

[2] A. Alsahly, J. Stascheit, G. Meschke, Advanced Finite Element Modeling of Excavation and Advancement Processes in Mechanized Tunneling. *Advances in Engineering Software*, **100**, 198–214, 2016.

[3] S. Freitag, B.T. Cao, J. Ninić, G. Meschke, Hybrid Surrogate Modelling for Mechanised Tunnelling Simulations with Uncertain Data. *International Journal of Reliability and Safety*,**9**(2/3), 154–173, 2015.

[4] S. Freitag, B.T. Cao, J. Ninić, G. Meschke, Recurrent Neural Networks and Proper Orthogonal Decomposition with Interval Data for Real-Time Predictions of Mechanised Tunnelling Processes. *Computers and Structures*, in review.

[5] M. Hanss, *Applied Fuzzy Arithmetic: An Introduction with Engineering Applications*, Springer, 2005.

[6] B. Möller, W. Graf, M. Beer, Fuzzy structural analysis using α-level optimization. *Computational Mechanics*, **26**(6), 547–565, 2000.

[7] D. Moens, D. Vandepitte, A survey of non-probabilistic uncertainty treatment in finite element analysis. *Computer Methods in Applied Mechanics and Engineering*, **194**, 1527–1555, 2005.

[8] B. Möller, U. Reuter, Prediction of uncertain structural responses using fuzzy time series. *Computers and Structures*, **86**, 1123–1139, 2008.

[9] S. Ferson, V. Kreinovich, L. Ginzburg, D.S. Myers, K. Sentz,
textit Constructing probability boxes and Dempster-Shafer structures, Tech. Rep. SAND2002-4015, Sandia National Laboratories, 2003.

[10] S. Freitag, W. Graf, M. Kaliske, Recurrent Neural Networks for Fuzzy Data. *Integrated Computer-Aided Engineering*, **18**(3), 265–280, 2011.

[11] A. Chatterjee, An Introduction to the Proper Orthogonal Decomposition. *Current Science*, **78**, 808–817, 2000.

[12] B.T. Cao, S. Freitag, G. Meschke, A Hybrid RNN-GPOD Surrogate Model for Real-time Settlement Predictions in Mechanised Tunnelling. *Advanced Modeling and Simulation in Engineering Sciences*, **3**, Paper 5, 1–22, 2016.

[13] R.L. Hardy, Theory and Applications of the Multiquadric-biharmonic Method: 20 Years of Discovery 1968-1988. *Computers & Mathematics with Applications*, **19**(8/9), 163–208, 1990.

[14] P. Paatero, U. Tapper, Positive Matrix Factorization: A Nonnegative Factor Model with Optimal Utilization of Error Estimates of Data Values. *Environmetrics*, **5**(2), 111–126, 1994.

[15] H. Kim, H. Park, Nonnegative Matrix Factorization Based on Alternating Nonnegativity Constrained Least Squares and Active Set Method. *SIAM Journal on Matrix Analysis and Applications*, **30**(2), 713–730, 2008.

[16] R. Everson, L. Sirovich, Karhunen-Loeve Procedure for Gappy Data. *Journal of the Optical Society of America A: Optics, Image Science and Vision*, **12**8, 1657–1664, 1995.