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UNCECOMP 2017
2nd ECCOMAS Thematic Conference on
Uncertainty Quantification in Computational Sciences and Engineering
M. Papadrakakis, V. Papadopoulos, G. Stefanou (eds.)
Rhodes Island, Greece, 15–17 June 2017

RANDOM VIBRATION ANALYSIS OF A SINGLE DEGREE OF FREEDOM UNDER IMPRECISE PROABILITY ASSIGNMENTS

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Keywords: Imprecise probability, random vibration, Miles' equation, bounds on failure probability, design of aircraft structural components.

Abstract. *The random vibration analysis of aircraft structural components is often performed with simplified techniques, such as Miles' Equation, which yields the root mean square of the acceleration response. The failure probability of the structural component is then established as the probability that the displacement exceeds a given threshold with a velocity in a certain period of time. Nonetheless, the dominant frequency of the system may be uncertain because of manufacturing variability, and because of variation in joints and connections of structural components. In this paper the conventional approach is extended in two ways: (i) firstly, the system parameters are taken to be uncertain rather than known, (ii) secondly, the uncertainty is modelled with an imprecise probability description, rather than with conventional pdfs (which usually are not known, due to a lack of empirical information). When the system parameters are modelled with a probability density function, the proposed approach yields: (a) the unconditional failure probability across an ensemble of Single Degree of Freedom (SDoF) systems; (b) the probability that the failure probability exceeds a specified target level across the ensemble of SDoFs. By introducing uncertainty in the probabilistic assignment of the system parameters, the bounds on these quantities are readily obtained. The feasibility of the proposed approaches are shown through their application to a structural panel of a spacecraft.*

doi: 10.7712/120217.5355.17233

1 INTRODUCTION

In the design of aircraft structural components is often assumed that only the randomness in the excitation (of mechanical and acoustic nature) is of concern, while the parameters of the structural components are assumed to be deterministic [1]. The random vibration analysis of aircraft structural components is often performed with simplified techniques, such as Miles' Equation, which assumes that a structural component, or a secondary structure, has a dominant natural frequency with respect to the structural response [1]. This equation yields the root mean square of the acceleration, as a function of the natural frequency of an equivalent Single Degree of Freedom (SDoF), of the quality factor and of the value of the random external force (expressed as a Power Spectral Density) at the natural frequency of the SDoF. The failure probability of the structural component is then established as the probability that the displacement exceeds a given threshold with a velocity in a certain period of time, by using well established techniques [1-3]. Nonetheless, the dominant frequency of the system may be uncertain because of manufacturing variability, and because of variation in joints and connections of structural components.

In this paper the conventional approach is extended in two ways: (i) firstly, the system parameters are taken to be uncertain rather than known, (ii) secondly, the uncertainty is modelled with an imprecise probability description [4], rather than with conventional pdfs (which usually are not known, due to a lack of empirical information). When the system parameters are modelled with a probability density function, the proposed approach yields: (a) the unconditional failure probability across an ensemble of SDoFs; (b) the probability that the failure probability exceeds a specified target level across the ensemble of SDoFs. By introducing uncertainty in the probabilistic assignment of the system parameters, the bounds on these quantities are readily obtained.

The standard random vibration analysis of aircraft structural components is reviewed in section 2. In section 3, the system parameters are taken to be uncertain rather than known, and described using a specified probability density function. Two methods are proposed: (i) Method A addresses the problem of estimating the probability that a deterministic limit value of the SDoF displacement response is exceeded by the response variable across the ensemble of SDoFs; (ii) Method B yields the probability that the failure probability itself exceeds a critical value. In section 4, an imprecise probability model based on a generalization of the Maximum Entropy distribution under uncertain statistical information is reviewed, and then Method A and B are extended to include this type of description to yield bounds on the two probability estimates. The random vibration analysis of a Single Degree of Freedom system with uncertainty in the system parameters, and in the probabilistic assignment of the system parameters is addressed in section 5.

2 REVIEW OF RANDOM VIBRATION ANALYSIS OF AIRCRAFT STRUCTURAL COMPONENTS

Miles' Equation [1] is a well-established technique used to assess the performance of aircraft structural components under random vibration. In this context it is assumed that a structural component, or a secondary structure, has a dominant natural frequency with respect to the structural response. The random vibration analysis of aircraft structural components reduces to the study of a Single Degree of Freedom (SDOF), consisting of a mass m , a spring

k and a damper c, subject to a random external force $F(t)$ which produces a displacement of the mass $y(t)$, as shown in Figure 1.

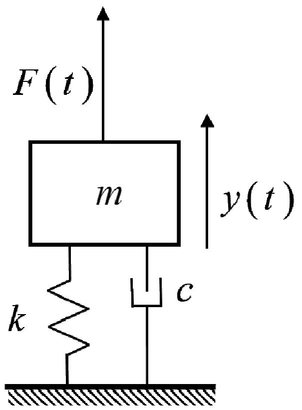

Figure 1: Damped SDoF subject to a force

The governing equation of the problem can be written as:

$$\ddot{y}(t) + 2\zeta\omega_n \dot{y}(t) + \omega_n y(t) = \frac{F(t)}{m} \tag{1}$$

being $\zeta = c/\left(2\sqrt{k\ m}\right)$ the damping ratio, and $\omega_n = \sqrt{k/m} = 2\pi \bar{f}_n$ the SDOF natural frequency corresponding to the dominant natural frequency of the structural component.

If the input is a broadband random loading with a constant spectral density over the frequency range of interest, $S_{FF}\left(\bar{f}\right) = S_0$ (with units of $\left(\mathrm{N}/kg\right)^2/\mathrm{Hz}$), being $\bar{f}$ the frequency expressed in Hz, the solution of the SDoF governing equation yields the Miles' Equation [1]:

$$\sigma_y = y_{rms} = \sqrt{\frac{QS_0\left(\bar{f}_n\right)}{32\pi^3 \bar{f}_n^{\,3}}} \tag{2}$$

This equation expresses the root mean square (rms) of the mass displacement, σ_y as a function of the natural frequency of the SDoF, $\bar{f}_n$, of the quality factor $Q = 1/(2\zeta) = \left(\sqrt{k\ m}\right)/c$ and of the value of the excitation force spectral density at the natural frequency $S_0\left(\bar{f}_n\right)$.

The rms of the mass displacement is then used to verify that the structure is be able to withstand a certain stress levels, such as the yield stress. This is achieved by defining the failure probability of the structural component, as the probability that the displacement $y(t)$ exceeds a given level b with a velocity $\dot{y}(t)$ in a certain period of time τ. It is well known that by assuming that each crossing of the barrier is independent from each other and randomly distributed along the time axis, the failure probability P_f can be expressed as [1-3]:

$$P_f = 1 - \exp\left[-\nu_b^+ \tau \right] \tag{3}$$

Being ν_b^+ the average number of positive crossing per unit of time of a barrier b, whose distribution is given by [1-3]:

$$\nu_b^+ = \frac{\sigma_{\dot{y}}}{2\pi\sigma_y} \exp\left[-\frac{1}{2}\left(\frac{b}{\sigma_y} \right)^2 \right] \tag{4}$$

Being $\sigma_{\dot{y}}$ the mean square of the velocity which is given by [1-3]:

$$\sigma_{\dot{y}} = \sqrt{\frac{S_0}{8\xi\left(2\pi\bar{f}_n\right)}} \tag{5}$$

The main aim of the analysis is therefore to establish the failure probability obtained with Equation (3) and compare it to a failure probability target. If the failure probability obtained is higher than the target one, a different design of the aircraft structural component should be explored. Nonetheless, the dominant frequency of the system may be uncertain, leading to uncertainty in the failure probability estimate. This conventional approach is extended in the next sections to account for the effect of the uncertainties in the system parameters.

3 RANDOM VIBRATION ANALYSIS OF AIRCRAFT STRUCTURAL COMPONENTS WITH UNCERTAIN SYSTEM PARAMETERS

Let us include uncertainty in the system parameters and model this uncertainty with a probability density function. The problem concern with an ensemble of SDoFs, where each element of the ensemble has a specified value of the natural frequency. When subject to the random loading, each element of the ensemble is characterized by a random displacement response. If the natural frequency of the SDOF $\bar{f}_n$ is described by a probability distribution $p\left(\bar{f}_n\right)$, two different approaches can be considered to yield two probability estimates which can be very useful from the engineering point of view.

3.1 Method A – unconditional failure probability

Method A enables the evaluation of the unconditional failure probability across an ensemble of SDoFs. This is expressed as:

$$\bar{P}_f = \int P_f\left(\bar{f}_n\right) p\left(\bar{f}_n\right) \mathrm{d}\bar{f}_n \tag{6}$$

Being $P_f\left(\bar{f}_n\right)$ the failure probability of a member n of the ensemble (for fixed $\bar{f}_n$) which is computed using Equation (3). This type of integral can be evaluated numerically by considering a grid of integration points (direct integration) or by employing Monte Carlo Simulations [5].

3.2 Method B – probability of failure probability exceeding a target level

Method B yields the probability that the failure probability exceeds a specified target level across the ensemble of SDoFs. This is expressed as:

$$P\left[P_f > \theta\right] = \int_{P_f > \theta} p\left(\bar{f}_n\right) \mathrm{d}\bar{f}_n \tag{7}$$

In most cases the analytical solution of this integral is not known and its approximate solution can be obtained by using numerical integration methods or Monte Carlo Simulations [5].

While Method A addresses the problem of estimating the probability that a deterministic limit value of the SDoF displacement response is exceeded by the response variable across the ensemble of SDoFs; Method B addresses a different type of problem: by specifying a critical value of the failure probability, it yields the probability that the failure probability itself exceeds this critical value. Therefore, these approaches yield two probability estimates which can be very useful from the engineering point of view.

4 INTRODUCING UNCERTAINTY IN THE PROBABILISTIC ASSIGNMENT OF THE SYSTEM PARAMETERS

4.1 Generalization of the Maximum Entropy distribution under uncertain statistical information

One of the main difficulty in performing a probabilistic uncertainty analysis is to specify an appropriate pdf of the uncertain system parameters. This might be caused by insufficient data required to empirically determine the pdf, because of cost or time constraints, or because the structure does not yet exist. Considering a wrong pdf might lead to overestimating or underestimating the system performance, therefore it is extremely important to represent the current state of knowledge using only the available information.

Often the only available information on the probabilistic assignment of some system parameters is that some statistical moments are known to lie in a certain domain (eg. the mean can take values between an upper and lower bounds, and/or there is a certain probability of finding the uncertain variable within certain bounds). For example, this domain can be defined as a set of inequality constraints on the statistical expectation:

$$v_{j,\min} \le v_j = \mathrm{E}\left[f_j\left(x\right)\right] = \int f_j\left(x\right) p\left(x\right) \mathrm{d}\,x \le v_{j,\max}, \qquad j = 2,3,...,n \tag{8}$$

where $v_{j,\min}$ and $v_{j,\max}$ are the lower and upper bound on the jth statistical expectation v_j and $f_j\left(x\right)$ is a specified functions of x. For example, if $f_j\left(x\right) = x$ then the constraints are specified on the mean value, alternatively if $f_j\left(x\right) = x^2$ they are specified on the second moment.

Many distributions belonging to the family of polynomial distributions, exponential distributions, maximum entropy distributions, and others can satisfy these inequality constraints. The principle of Maximum Entropy [6] allows the estimation of a subjective pdf $p\left(x\right)$ of an uncertain variable x which best represent the current state of knowledge by maximizing the relative entropy subject to constraints representing the available information [6]. However, if Maximum Entropy is used to find the distribution with the largest entropy whose statistical moments lie within the domain, the information at only a single point in the domain would be used and other information would be discarded. Recently a generalisation of this approach has been proposed [4], that first constructs a family of maximum entropy distributions consistent with the statistical inequality constraints, then propagates the family of pdfs through the equations describing the problem on hand in order to yield the pdf which maximizes (or equivalently minimizes) a specified engineering metric [4].

With this approach [4], the pdf of a vaguely known variable x is expressed as the exponential of a linear combination of specified functions of the random variable $f_j(x)$ and some bounded variables $\mathbf{a}$ (referred to here as basic variables), so that:

$$p(x|\mathbf{a}\in R) = t(x)\exp\left[\sum_{j=1}^{n} a_j f_j(x)\right]. \tag{9}$$

The basic variables are contained in the vector $\mathbf{a}$, which has entries a_j with $j = 2,3...,n$ that lie within an admissible region R (which can be an interval, a convex region, etc.). These basic variables substitute the Lagrange multipliers of a standard Maximum Entropy distribution and are such that they can have any possible pdf within certain bounds, including the extreme case of a delta function at any point between the bounds. Therefore, Eq. (9) represents a family of maximum entropy distributions defined over the set of basic variables $\mathbf{a}$. If a parameter is not "basic", then its pdf can be expressed in terms of the basic parameters, and thus only this type of parameter is considered in what follows. The term $f_1(x) = 1$, the coefficient a_1 is dependent on the basic variables a_j (with $j = 2,3...,n$) and it is chosen to satisfy the normalisation condition. The function $t(x)$ is the reference pdf introduced to allow the pdf to be frame invariant. In the forthcoming discussions, without loss of generality, it will be assumed that $t(x) = 1$.

The construction of the family of Maximum Entropy distributions consists of the following steps:

1. Express the pdf of a vaguely known variable in the form of Eq. (9).
2. Convert the statistical moments domain (*m*-domain) into a basic variable domain (so-called *a*-domain) [4].
3. Subdivide the *a*-domain into a sufficient number of grid of points.
4. Compute the coefficient a_1 for each point of the grid (by using the normalisation condition) to derive the corresponding pdf.

The mapping of the *m*-domain to the *a*-domain (Step 2) can be very computationally demanding even for a simple domain, since it requires the solution of a set of non-linear equations for each point that need to be mapped, and convergence problems may occur.
In reference [4] an approach for computing an approximate mapping has been presented. This approach is based on expressing the variation of the pdf over the *m*-domain as a Taylor series, centered at $\mathbf{a}^*$ which is a point of the *a*-domain obtained from the mapping of the mid-point

of the m-domain (denoted by $\mathbf{v}^*$), so that:

$$v_s(\mathbf{a}) = c_s^1(\mathbf{a}^*) - \sum_{j=2}^{n}(a_j - a_j^*)c_{s,j}^2(\mathbf{a}^*) - \frac{1}{2}\sum_{j=2}^{n}\sum_{k=2}^{n}(a_j - a_j^*)(a_k - a_k^*)c_{s,j,k}^3(\mathbf{a}^*) \qquad s = 1,2,\dots n \tag{10}$$

where $c_s^1(\mathbf{a}^*), c_{s,j}^2(\mathbf{a}^*)$ and $c_{s,j,k}^3(\mathbf{a}^*)$ are the first, second and third order cumulants, respectively, calculated at the $\mathbf{a}^*$ point. The term $c_s^1(\mathbf{a}^*)$ corresponds to the statistical expectation calculated at $\mathbf{a}^*$, therefore it is equivalent to the term v_s^* which can be obtained directly from the m-domain, while $c_{s,j}^2(\mathbf{a}^*)$ and $c_{s,j,k}^3(\mathbf{a}^*)$ are calculated by numerical integration of:

$$c_{s,j}^2(\mathbf{a}^*) = \int (f_s(x) - v_s^*)(f_j(x) - v_j^*)\, p(x|\mathbf{a}^*)\,\mathrm{d}x, \tag{11}$$

$$c_{s,j,k}^3(\mathbf{a}^*) = \int (f_s(x) - v_s^*)(f_j(x) - v_j^*)(f_k(x) - v_k^*)\, p(x|\mathbf{a}^*)\,\mathrm{d}x. \tag{12}$$

Each point of the a-domain $\mathbf{a}$ corresponding to a point $v(\mathbf{a})$ of the m-domain can be then computed by solving a set of n (where n corresponds to the domain dimension) equations (in the form of Eq. (10)) where a linear or quadratic approximation might be considered. High order terms may also be included [4]. Therefore, if r points of a 3-dimensional m-domain are mapped into the a-domain, this would require solving r sets of 3 equations.

By adopting this approach, the edges of the m-domain can be mapped to the a-domain in an efficient way, thus allowing a permissible region of the a-domain to be determined.

4.2 Extension of Method A and Method B

The uncertainty in the probabilistic assignment of the system parameters of the SDOF is now expressed as in Equation (9), so that the pdf of the natural frequency of the system is given by $p(\bar{f}_n|\mathbf{a})$. For fixed basic variables $P_f(\bar{f}_n|\mathbf{a})$ is then obtained using Eq. (3), then Equation (6) can be generalized to yield the unconditional failure probability for fixed basic variables:

$$\bar{P}_f(\mathbf{a}) = \int P_f(\bar{f}_n|\mathbf{a})\, p(\bar{f}_n|\mathbf{a})\,\mathrm{d}\bar{f}_n \tag{13}$$

As a result the upper and lower bounds on the failure probability are given by:

$$\min_{\mathbf{a}\in R}\left(\bar{P}_f(\mathbf{a})\right) \le \bar{P}_f \le \max_{\mathbf{a}\in R}\left(\bar{P}_f(\mathbf{a})\right) \tag{14}$$

Similarly, Method B can be generalized to yield the probability that the failure probability exceed a certain threshold θ for fixed basic variables $\mathbf{a}$:

$$P\left[(P_f > \theta)|\mathbf{a}\right] = \int_{P_f > \theta} p(\bar{f}_n|\mathbf{a})\,\mathrm{d}\bar{f}_n \tag{15}$$

To yield the following bounds:

$$\min_{\mathbf{a}\in R}\left(P\left[\left(P_f>\theta\right)\middle|\mathbf{a}\right]\right)\le P\left[\left(P_f>\theta\right)\right]\le\max_{\mathbf{a}\in R}\left(P\left[\left(P_f>\theta\right)\middle|\mathbf{a}\right]\right) \tag{16}$$

5 NUMERICAL APPLICATION

Let us consider a structural panel of a spacecraft with fundamental frequency of $\bar{f}_n$ =250 Hz, damping ratio $\zeta = 0.05$ (quality factor $Q = 10$), subject to a white noise Power Spectral Density function $S_0 = 100$ $(\mathrm{N}/kg)^2/\mathrm{Hz}$. A displacement threshold level $b = 7\times\sigma_y$ =0.0018 mm (σ_y being computed with Eq. (2)) during a time interval $\tau = 3$ hours is specified. Using the standard approach (Eq. (3)), the resulting failure probability is: $P_f = 6.1\times10^{-5}$.

Now, let us consider that the fundamental frequency of the panel is uncertain since the spring stiffness is uncertain. Specifically, let us consider the case that he following information are available on the spring stiffness: (i) the variable is positive (ii) the vertices of a convex region of the statistical expectations $\mathrm{E}[x]$ and $\mathrm{E}[\ln(x)]$ are known, and as shown in Figure 2.

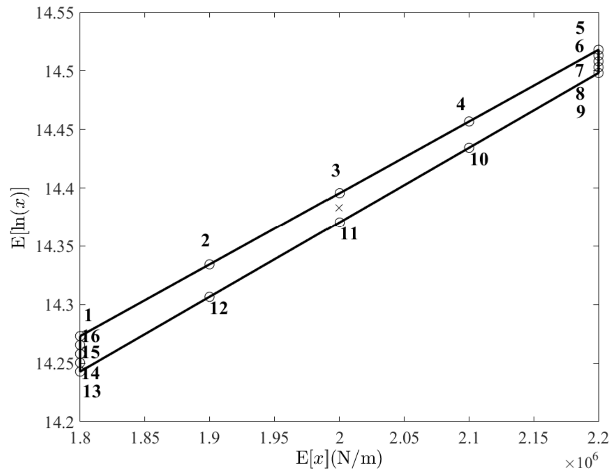

Figure 2: Statistical expectation domain (m-domain). The circles indicate points along the m-domain, while the "x" indicates the middle point of the m-domain

The vertices of the domain have the following coordinates: point 1 $(18.0\times10^5, 14.273)$; point 5 $(22.0\times10^5, 14.518)$; point 9 $(22.0\times10^5, 14.498)$; and point 13 $(18.0\times10^5, 14.243)$.

Following the procedure described in subsection 4.1, the pdf of x is written in the same form of Equation (9), that is a Gamma distribution:

$$p(x|\mathbf{a}) = \exp[a_1 - a_2 x - a_3 \ln(x)], \tag{17}$$

where a_2 and a_3 are the two unknown basic variables, while a_1 is obtained by applying the normalisation condition (for $\mathrm{Re}[a_2] > 0; \mathrm{Re}[a_3] < 1$) as:

$$a_1 = -\ln\left(a_2^{(a_3-1)}\Gamma(1-a_3)\right), \tag{18}$$

where $\Gamma(\bullet)$ is the gamma function.
Although the *m*-domain is characterized by linear surfaces, the corresponding *a*-domain is not; 16 points along the *m*-domain are mapped onto the *a*-domain by using the strategy summarized in section 4.1. The approximate mapping strategy would consider a Taylor series expansion over the whole domain, taking the mid-point of the domain as the reference point for the expansion. The mid-point has coordinates $E[x^*] = 20.0 \times 10^5$ N/m and $E[\ln(x)^*] = 14.383$, and it is indicated with an "x" in Figure 2, and its mapping to the *a*-domain corresponds to $\mathbf{a}^* = (20.696 \times 10^5, -3.139)$. Each point in the *a*-domain obtained with the approximate procedure has been verified by solving a set of non-linear equations showing a good agreement. The *a*-domain so obtained is shown in Figure 3.

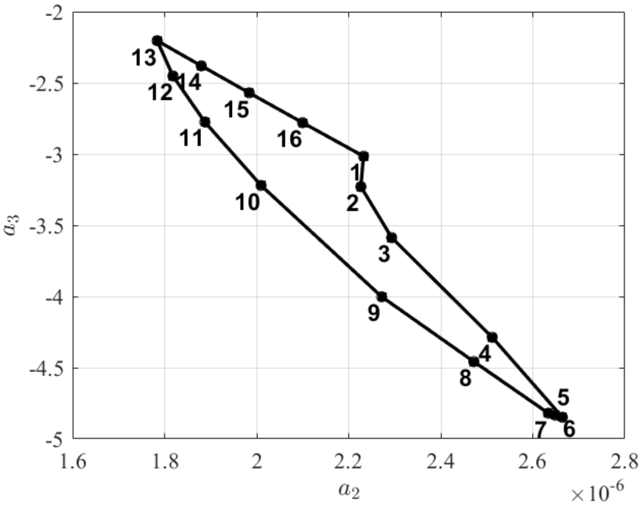

Figure 3: Basic variable domain (a-domain)

A set of 430 pdfs was generated by considering the 414 point inside the *a*-domain (considering a grid of 50×50 equally-spaced points overlaying the *a*-domain) and the 16 points along the *a*-domain.

The Maximum Entropy distribution, which is the gamma pdf with the largest entropy that satisfies the statistical moment conditions, can be computed by calculating the entropy (measured in nats) associated with each gamma distribution:

$$h_e = \ln\left(\frac{\Gamma(1-a_3)}{a_2}\right) + a_3\Psi(1-a_3) + 1 - a_3 \tag{19}$$

and then searching in the *a*-domain, the combination of basic variables which maximise Eq. (19). For the case under investigation, the pdf with the largest entropy is the one with basic variables corresponding to point 10 of the *a*-domain.

By performing a transformation of variables, the pdf of the natural frequency conditional on the basic variables, $p(\overline{f}_n|\mathbf{a})$, is obtained as:

$$p(\overline{f}_n|\mathbf{a}) = 2\gamma\overline{f}_n \exp\left[a_1 - a_2\gamma\overline{f}_n^{\,2} - a_3\ln\left(\gamma\overline{f}_n^{\,2}\right)\right], \tag{20}$$

where $\gamma = 4\pi^2 m$ with m= 0.81 kg, and a_1 is obtained by applying the normalisation condition (for $\mathrm{Re}[a_2] > 0; \mathrm{Re}[a_3] < 0$) as:

$$a_1 = \ln\left(\frac{a_2 \gamma^{a_3} (a_2 \gamma)^{-a_3}}{\Gamma(1 - a_3)} \right). \tag{21}$$

The maximum entropy distribution (obtained at point 10 of the domain) would yield a failure probability of: $\bar{P}_f |_{\text{point 10}} = 0.2935$ (solving Eq. (6) using direct integration), that is much larger than the result obtained using a deterministic value. This results can be explained by considering the sharp variation of the P_f close to 250 Hz, as shown in Figure 4.

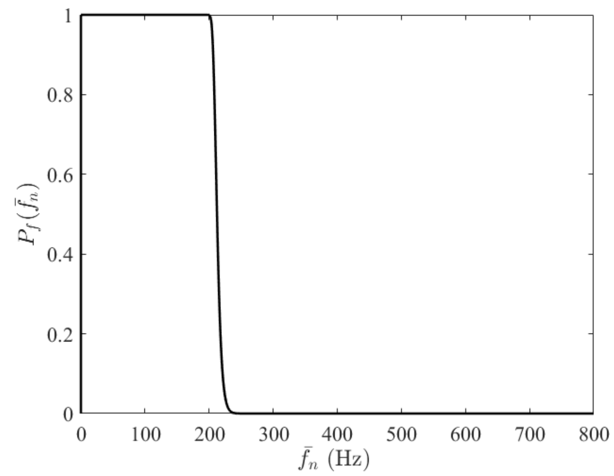

Figure 4: P_f as a function of frequency

If we now set a failure probability target $\theta = 0.001$, the probability of exceedance of this target (obtained solving Eq. (7) using numerical integration) for the MaxEnt distribution is $P\left[\left(P_f > \theta\right)\right]\Big|_{\text{point 10}} = 0.4605$.

By letting varying the basic variable within the *a*-domain (using the 430 distributions), it has been found that the unconditional failure probability can take values between $0.2126 \le \bar{P}_f \le 0.4303$ (obtained at points 7 and 13, respectively); therefore the MaxEnt pdf would underestimate the upper bound.
Similarly, the probability that the failure probability target is exceeded can vary from $0.3876 \le P\left[\left(P_f > \theta = 0.001\right)\right] \le 0.5918$ (obtained at points 7 and 13, respectively); also in this case the MaxEnt pdf would underestimate the response upper bound.

It can be concluded that employing the proposed uncertainty model enable an enhanced description of structural panel performance yielding the maximum and minimum values that the engineering metric of interest (eg the failure probability) might take based on the available information, rather than yielding a single value which might significantly underestimate or overestimate the engineering metric of interest.

6 CONCLUSIONS

Two approaches for the random vibration analysis of a single degree of freedom under uncertainty in the system parameters have been presented. These two approaches yield: (i) the probability that a deterministic limit value of the SDoF displacement response is exceeded by the response variable across the ensemble of SDoFs; and the (ii) probability that the failure probability itself exceeds a critical value. These two probability estimates can be very useful from the engineering point of view, and in particular for the design of aircraft structural components which are usually performed assuming deterministic system parameters.

These two approaches have been also extended to account for the uncertainty in the probabilistic assignment of the system parameters, yielding bounds on the two probability estimates. Through a numerical application, it has been shown that including uncertainty in the probabilistic assignment lead to an enhanced description of the system response that support the development of a design solution which can be more robust with respect to the design constraints under uncertainties in the models parameters.

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