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# RESPONSE SENSITIVITY OF STRUCTURAL SYSTEMS SUBJECTED TO FULLY NON-STATIONARY RANDOM PROCESSES

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**Abstract**

*A method for the evaluation of the statistics of response sensitivity of both classically and non-classically damped discrete linear structural systems under fully non-stationary stochastic seismic processes is presented. To do this the evolutionary frequency response function, also referred in literature as the time-frequency varying response function, plays a central role in the evaluation of the spectral characteristics of non-stationary response.*

*The proposed approach requires the following items: a) to write governing motion equations in state-variables, which are very suitable to evaluate the statistics of the response of both classically and non-classically damped discrete linear structural systems by an unified approach; b) to evaluate in explicit closed form solutions the derivatives of time-frequency response vector functions with respect to the parameters that define the modified structural model; c) to obtain the sensitivity of the structural response statistics by frequency domain integrals.*

*A numerical application shows that the proposed approach is suitable to cope with practical problems of engineering interest.*

**Keywords:** Sensitivity analysis, Fully non-stationary processes, Non-geometric spectral moments, Evolutionary power spectral density function; Evolutionary frequency response function.


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## 1 INTRODUCTION

During the analysis of structural systems, the reference structural parameters could be modified for design reasons. This is very frequent in optimization procedures, design of devices for vibrations control, etc. (see e.g., [1,2,3]). In this framework, the sensitivity analysis (i.e. the evaluation of partial derivatives of a performance measure with respect to system parameters) is a suitable vehicle to evaluate the response variation of structures under the influence of changes of parameter values.

Strong motion earthquakes are certainly the main critical actions for structures located in the seismically active regions of the earth. The analysis of recorded accelerograms in different sites shows that earthquake ground motion time-histories are non-stationary processes in both amplitude and frequency content. Then, the stationary models fail to reproduce the time-varying intensity, which is typical of real earthquakes ground-motion accelerograms. To take into account the time variability, the so-called *quasi-stationary* (or *uniformly modulated non-stationary*) random processes have been introduced [see e.g. 4,5]. These processes are constructed modulating the amplitude of a stationary zero-mean Gaussian random process through a deterministic function of time; for this reason they are also called *separable non-stationary stochastic processes*. However, these processes catch only the time-varying intensity of the accelerograms. To consider simultaneously both the amplitude and frequency changes, time-frequency varying deterministic modulating functions have been introduced in the characterization of the seismic process. The latter processes are referred as *fully or non-separable non-stationary stochastic processes* (see e.g., [6,7]).

Several papers have been devoted afterward to study the sensitivity of the response of structural systems subjected to stochastic excitations. As an example, Szopa [8] studied the stochastic sensitivity of the Van der Pol equation. Benfratello et al. [9] proposed a procedure, in the time domain, to evaluate the sensitivity of the statistical moments of the response of structural systems for stationary Gaussian and non-Gaussian white input processes. Proppe et al. [10] showed that the sensitivity analysis can be considered as an application of the Equivalent Linearization for design problem. Chaudhuri and Chakraborty [11] dealt with the response sensitivity evaluation in the frequency domain of structures subjected to non-stationary seismic processes. In Cacciola et al. [12] the sensitivities governing the evolution of spectral moments of the response are evaluated by solving set of differential equations once the Kronecker algebra is applied.

For linear structural systems subjected to non-stationary stochastic excitations, the *evolutionary frequency response function*, also referred in literature as the *time-frequency varying response function*, plays a central role in the evaluation of the statistics of the response [13]. In fact, by means of this function, it is possible to evaluate in explicit form the *evolutionary power spectral density* of the response and, consequently, *the non-geometric spectral moments*, which are required in the prediction of the safety of structural systems subjected to non-stationary random excitations (see e.g., [14-19]).

In recent studies [20,21], the senior authors, have evaluated in explicit form, for both classically and non-classically damped structural systems, the *time-frequency varying response function*.

In this study handy expressions for the sensitivities of *non-geometric spectral moments* of the structural response of linear classically or non-classically damped linear structural systems subjected to both separable and non-separable non-stationary excitations are evaluated. The proposed approach requires the following items: a) to determine sensitivities of *evolutionary frequency response functions* by means of explicit closed form solutions; b) to evaluate the sensitivity of the structural response statistics by frequency domain integrals.

A numerical application shows that the proposed approach is suitable to cope with practical problems of engineering interest.

## 2 DYNAMIC RESPONSE SENSITIVITIES FOR DETERMINISTIC LOADS

The sensitivity analysis consist in the evaluation of the change in the system response due to system parameter variations in the neighborhood of prefixed values, called “nominal parameter”. To this aim, preliminarily the set of significant parameters, for which the influence on the response has to be evaluated, are collected in the *r*-component vector $\boldsymbol{\alpha}$, where *r* being the number of the significant parameters taken into account. For a quiescent structural system at time $t = t_0$, the dependence of the damping and stiffness matrices of the structure, and of the response vector collecting the nodal displacements, on the actual value $\boldsymbol{\alpha}$ of the significant parameter vector, is expressed as:

$$\mathbf{M}\,\ddot{\mathbf{U}}(\boldsymbol{\alpha},t)+\mathbf{C}(\boldsymbol{\alpha})\dot{\mathbf{U}}(\boldsymbol{\alpha},t)+\mathbf{K}(\boldsymbol{\alpha})\mathbf{U}(\boldsymbol{\alpha},t)=-\mathbf{M}\,\boldsymbol{\tau}F(t);\quad \mathbf{U}(\boldsymbol{\alpha},t_0)=\mathbf{0} \tag{1}$$

where $\mathbf{M}$, $\mathbf{C}(\boldsymbol{\alpha})$, and $\mathbf{K}(\boldsymbol{\alpha})$ are the $n\times n$ mass, damping, and stiffness matrices of the structure, $\mathbf{U}(\boldsymbol{\alpha},t)$ is the *n*-dimensional vector of nodal displacements relative to the ground, $\boldsymbol{\tau}$ is the *n*-dimensional array listing the influence coefficients of the ground shaking, $F(t)$ is the time-dependent loading vector, and a dot over a variable denotes differentiation with respect to time.

Denoting with $\boldsymbol{\alpha}_0$ the vector of the significant parameters in correspondence of the nominal parameters, any vector $\boldsymbol{\alpha}$ in the neighborhood of $\boldsymbol{\alpha}_0$ can be represented as:

$$\boldsymbol{\alpha}=\boldsymbol{\alpha}_0+\Delta\boldsymbol{\alpha}\,, \tag{2}$$

where $\Delta\boldsymbol{\alpha}$ is assumed to be a vector collecting small parameter variations with respect to the nominal parameter vector $\boldsymbol{\alpha}_0$. In order to evaluate the response sensitivity, the equation of motion (1) is written as:

$$\mathbf{M}\,\ddot{\mathbf{U}}(\boldsymbol{\alpha},t)+\left[\mathbf{C}(\boldsymbol{\alpha}_0)+\Delta\mathbf{C}(\boldsymbol{\alpha})\right]\dot{\mathbf{U}}(\boldsymbol{\alpha},t)+\left[\mathbf{K}(\boldsymbol{\alpha}_0)+\Delta\mathbf{K}(\boldsymbol{\alpha})\right]\mathbf{U}(\boldsymbol{\alpha},t)=-\mathbf{M}\,\boldsymbol{\tau}F(t);\quad \mathbf{U}(\boldsymbol{\alpha},t_0)=\mathbf{0} \tag{3}$$

in which $\mathbf{K}(\boldsymbol{\alpha}_0)$ and $\mathbf{C}(\boldsymbol{\alpha}_0)$ are the stiffness and damping matrices of the structure evaluated in correspondence of the nominal parameter vector $\boldsymbol{\alpha}_0$, while, $\Delta\mathbf{C}(\boldsymbol{\alpha})=\mathbf{C}(\boldsymbol{\alpha})-\mathbf{C}(\boldsymbol{\alpha}_0)$ and $\Delta\mathbf{K}(\boldsymbol{\alpha})=\mathbf{K}(\boldsymbol{\alpha})-\mathbf{K}(\boldsymbol{\alpha}_0)$. It follows that the structural system is non-classically damped. To solve Eq.(1) the equations of motion have to be rewritten in state variables:

$$\dot{\mathbf{Z}}(\boldsymbol{\alpha},t)=\mathbf{D}(\boldsymbol{\alpha})\mathbf{Z}(\boldsymbol{\alpha},t)+\mathbf{w}F(t);\quad \mathbf{Z}(\boldsymbol{\alpha},t_0)=\mathbf{0} \tag{4}$$

where $\mathbf{Z}(\boldsymbol{\alpha},t)$ is the $2n$-state vector variable while the $2n\times 2n$ matrix $\mathbf{D}(\boldsymbol{\alpha})$ and the $2n$-vector $\mathbf{w}$ are defined as:

$$\mathbf{Z}(\boldsymbol{\alpha},t)=\begin{bmatrix}\mathbf{U}(\boldsymbol{\alpha},t)\\ \dot{\mathbf{U}}(\boldsymbol{\alpha},t)\end{bmatrix};\quad \mathbf{D}(\boldsymbol{\alpha})=\begin{bmatrix}\mathbf{O}_{n,n} & \mathbf{I}_n\\ -\mathbf{M}^{-1}\mathbf{K}(\boldsymbol{\alpha}) & -\mathbf{M}^{-1}\mathbf{C}(\boldsymbol{\alpha})\end{bmatrix};\quad \mathbf{w}=\begin{bmatrix}\mathbf{0}_n\\ -\boldsymbol{\tau}\end{bmatrix}; \tag{5}$$

where $\mathbf{I}_n$ and $\mathbf{O}_{n,n}$ are respectively the identity and the zero matrices of $n\times n$ order while $\mathbf{0}_n$ stands for a $n$-dimensional vector. In order to evaluate the structural response the $2n\times 2n$

transition matrix $\boldsymbol{\Theta}(\boldsymbol{\alpha},t)$ has to be introduced [22,23], and for non-classically damped systems this matrix can be evaluated as:

$$\boldsymbol{\Theta}(\boldsymbol{\alpha},t)=\exp\left[\mathbf{D}(\boldsymbol{\alpha})t\right]=\boldsymbol{\Psi}(\boldsymbol{\alpha})\exp\left[\boldsymbol{\Lambda}(\boldsymbol{\alpha})t\right]\boldsymbol{\Psi}^{T}(\boldsymbol{\alpha})\mathbf{A}(\boldsymbol{\alpha})\equiv\boldsymbol{\Psi}^{*}(\boldsymbol{\alpha})\exp\left[\boldsymbol{\Lambda}^{*}(\boldsymbol{\alpha})t\right]\boldsymbol{\Psi}^{*T}(\boldsymbol{\alpha})\mathbf{A}(\boldsymbol{\alpha}) \tag{6}$$

in which $\mathbf{D}(\boldsymbol{\alpha})$ has been defined in Eq.(5), $\boldsymbol{\Lambda}(\boldsymbol{\alpha})$ and $\boldsymbol{\Psi}(\boldsymbol{\alpha})$ are the complex matrices collecting eigenvalues and eigenvectors respectively, depending of uncertain parameters $\boldsymbol{\alpha}$. Formally, these matrices can be evaluated by applying the *complex modal analysis*. According to this analysis the following coordinate transformation is introduced:

$$\mathbf{Z}(\boldsymbol{\alpha},t)=\boldsymbol{\Psi}(\boldsymbol{\alpha})\mathbf{X}(\boldsymbol{\alpha},t). \tag{7}$$

If $m$ is the number of modes selected for the analysis, $\mathbf{X}(\boldsymbol{\alpha},t)$ is a complex vector of order $2m$ and the complex matrix $\boldsymbol{\Psi}(\boldsymbol{\alpha})$, of order $(2n\times 2m)$, collects the complex eigenvectors, solutions of the following eigenproblem:

$$\mathbf{D}^{-1}(\boldsymbol{\alpha})\boldsymbol{\Psi}(\boldsymbol{\alpha})=\boldsymbol{\Psi}(\boldsymbol{\alpha})\boldsymbol{\Lambda}^{-1}(\boldsymbol{\alpha});\quad \boldsymbol{\Psi}^{T}(\boldsymbol{\alpha})\mathbf{A}(\boldsymbol{\alpha})\boldsymbol{\Psi}(\boldsymbol{\alpha})=\mathbf{I}_{2m} \tag{8}$$

where the superscript $T$ denotes the transpose operator, $\boldsymbol{\Lambda}$ is the diagonal matrix collecting the $2m$ complex eigenvalues and

$$\mathbf{A}(\boldsymbol{\alpha})=\begin{bmatrix}\mathbf{C}(\boldsymbol{\alpha}) & \mathbf{M}\\ \mathbf{M} & \mathbf{O}_{n,n}\end{bmatrix}. \tag{9}$$

In order to evaluate the first-order sensitivity, Eq.(4) must be differentiated with respect to $\boldsymbol{\alpha}$, setting $\boldsymbol{\alpha}=\boldsymbol{\alpha}_0$, leading to the following differential equation [12]:

$$\dot{\mathbf{s}}_{\mathbf{Z},i}(\boldsymbol{\alpha}_0,t)=\mathbf{D}(\boldsymbol{\alpha}_0)\mathbf{s}_{\mathbf{Z},i}(\boldsymbol{\alpha}_0,t)+\bar{\mathbf{F}}(\boldsymbol{\alpha}_0,t);\quad \mathbf{s}_{\mathbf{Z},i}(\boldsymbol{\alpha}_0,t_0)=\mathbf{0} \tag{10}$$

where the pseudo-force vector $\bar{\mathbf{F}}(\boldsymbol{\alpha}_0,t)$ is given by the equation

$$\bar{\mathbf{F}}(\boldsymbol{\alpha}_0,t)=\mathbf{D}'_i(\boldsymbol{\alpha}_0)\mathbf{Z}(\boldsymbol{\alpha}_0,t) \tag{11}$$

in which all the quantities are known. In Eq.(11) the matrix $\mathbf{D}'_i(\boldsymbol{\alpha}_0)$ can be readily determined deriving the matrix $\mathbf{D}(\boldsymbol{\alpha})$ with respect to $i$-th significant parameter $\alpha_i$. That is,

$$\mathbf{s}_{\mathbf{Z},i}(\boldsymbol{\alpha}_0,t)=\left.\frac{\partial\mathbf{Z}(\boldsymbol{\alpha},t)}{\partial\alpha_i}\right|_{\boldsymbol{\alpha}=\boldsymbol{\alpha}_0};\quad \mathbf{D}'_i(\boldsymbol{\alpha}_0)=\left.\frac{\partial}{\partial\alpha_i}\mathbf{D}(\boldsymbol{\alpha})\right|_{\boldsymbol{\alpha}=\boldsymbol{\alpha}_0}=\begin{bmatrix}\mathbf{O}_{n,n} & \mathbf{O}_{n,n}\\ -\mathbf{M}^{-1}\mathbf{K}'_i(\boldsymbol{\alpha}_0) & -\mathbf{M}^{-1}\mathbf{C}'_i(\boldsymbol{\alpha}_0)\end{bmatrix} \tag{12}$$

where

$$\mathbf{K}'_i(\boldsymbol{\alpha}_0)=\left.\frac{\partial}{\partial\alpha_i}\mathbf{K}(\boldsymbol{\alpha})\right|_{\boldsymbol{\alpha}=\boldsymbol{\alpha}_0};\quad \mathbf{C}'_i(\boldsymbol{\alpha}_0)=\left.\frac{\partial}{\partial\alpha_i}\mathbf{C}(\boldsymbol{\alpha})\right|_{\boldsymbol{\alpha}=\boldsymbol{\alpha}_0}. \tag{13}$$

It is noted that the set of first-order ordinary differential in Eq.(10) is formally similar to Eq.(4), which represents the equation of motion of the structural system in the state variable space. This means that the derivatives of the response with respect to the $i$-th parameter can be calculated by means of the same procedures used for response evaluation, that is:

$$\mathbf{s}_{\mathbf{Z},i}\left(\boldsymbol{\alpha}_0,t\right)=\boldsymbol{\Psi}\left(\boldsymbol{\alpha}_0\right)\int_{t_0}^{t}\exp\left[\boldsymbol{\Lambda}\left(\boldsymbol{\alpha}_0,t-\tau\right)\right]\boldsymbol{\Psi}^T\left(\boldsymbol{\alpha}_0\right)\mathbf{A}\left(\boldsymbol{\alpha}_0\right)\overline{\mathbf{F}}\left(\boldsymbol{\alpha}_0,\tau\right)\mathrm{d}\tau \tag{14}$$

It follows that the sensitivity vector of the response in state variables can be evaluated as:

$$\begin{aligned}\mathbf{s}_{\mathbf{Z},i}\left(\boldsymbol{\alpha}_0,t\right)&=\boldsymbol{\Psi}\left(\boldsymbol{\alpha}_0\right)\int_{t_0}^{t}\exp\left[\boldsymbol{\Lambda}\left(\boldsymbol{\alpha}_0,t-\tau\right)\right]\boldsymbol{\Psi}^T\left(\boldsymbol{\alpha}_0\right)\mathbf{A}\left(\boldsymbol{\alpha}_0\right)\mathbf{D}_i'\left(\boldsymbol{\alpha}_0\right)\mathbf{Z}\left(\boldsymbol{\alpha}_0,\tau\right)\mathrm{d}\tau\\&=\boldsymbol{\Psi}\left(\boldsymbol{\alpha}_0\right)\int_{t_0}^{t}\left\{\exp\left[\boldsymbol{\Lambda}\left(\boldsymbol{\alpha}_0,t-\tau\right)\right]\mathbf{B}_i\left(\boldsymbol{\alpha}_0\right)\left[\int_{t_0}^{\tau}\exp\left[\boldsymbol{\Lambda}\left(\boldsymbol{\alpha}_0,\tau-\rho\right)\right]F\left(\rho\right)\mathrm{d}\rho\right]\right\}\mathrm{d}\tau\,\mathbf{v}\left(\boldsymbol{\alpha}_0\right)\end{aligned} \tag{15}$$

where

$$\mathbf{Z}\left(\boldsymbol{\alpha}_0,\tau\right)=\boldsymbol{\Psi}\left(\boldsymbol{\alpha}_0\right)\left[\int_{t_0}^{\tau}\exp\left[\boldsymbol{\Lambda}\left(\boldsymbol{\alpha}_0,\tau-\rho\right)\right]F\left(\rho\right)\mathrm{d}\rho\right]\mathbf{v}\left(\boldsymbol{\alpha}_0\right) \tag{16}$$

and

$$\mathbf{v}(\boldsymbol{\alpha})=\boldsymbol{\Psi}^T(\boldsymbol{\alpha})\mathbf{A}(\boldsymbol{\alpha})\mathbf{w};\quad \mathbf{B}_i\left(\boldsymbol{\alpha}_0\right)=\boldsymbol{\Psi}^T\left(\boldsymbol{\alpha}_0\right)\mathbf{A}\left(\boldsymbol{\alpha}_0\right)\mathbf{D}_i'\left(\boldsymbol{\alpha}_0\right)\boldsymbol{\Psi}\left(\boldsymbol{\alpha}_0\right). \tag{17}$$

For deterministic excitation the sensitivity of the response can be evaluated by a step-by-step procedure [12,22,23].

## 3 DYNAMIC RESPONSE SENSITIVITY FOR FULLY NON-STATIONARY STOCHASTIC LOAD PROCESSES

### 3.1 Closed form solutions for the time-frequency varying response vector function

In the framework of non-stationary analysis of structures, the *spectral moments* can be evaluated in compact form by introducing the *pre-envelope covariance* (*PEC*) matrix. This matrix, in nodal space, is a $2n\times 2n$ Hermitian matrix, that, for non-classically damped systems, can be evaluated formally as [18,19]:

$$\begin{aligned}\boldsymbol{\Sigma}_{\mathbf{ZZ}}(\boldsymbol{\alpha},t)=\mathrm{E}\left\langle\mathbf{Z}(\boldsymbol{\alpha},t)\mathbf{Z}^{*T}(\boldsymbol{\alpha},t)\right\rangle&=\begin{bmatrix}\mathrm{E}\left\langle\mathbf{U}(\boldsymbol{\alpha},t)\mathbf{U}^{*T}(\boldsymbol{\alpha},t)\right\rangle & \mathrm{E}\left\langle\mathbf{U}(\boldsymbol{\alpha},t)\dot{\mathbf{U}}^{*T}(\boldsymbol{\alpha},t)\right\rangle\\ \mathrm{E}\left\langle\dot{\mathbf{U}}(\boldsymbol{\alpha},t)\mathbf{U}^{*T}(\boldsymbol{\alpha},t)\right\rangle & \mathrm{E}\left\langle\dot{\mathbf{U}}(\boldsymbol{\alpha},t)\dot{\mathbf{U}}^{*T}(\boldsymbol{\alpha},t)\right\rangle\end{bmatrix}\\&=\begin{bmatrix}\boldsymbol{\Lambda}_{0,\mathbf{UU}}(\boldsymbol{\alpha},t) & \mathrm{i}\boldsymbol{\Lambda}_{1,\mathbf{UU}}(\boldsymbol{\alpha},t)\\ -\mathrm{i}\boldsymbol{\Lambda}_{1,\mathbf{UU}}^{*T}(\boldsymbol{\alpha},t) & \boldsymbol{\Lambda}_{2,\mathbf{UU}}(\boldsymbol{\alpha},t)\end{bmatrix}\end{aligned} \tag{18}$$

where $\mathbf{Z}(\boldsymbol{\alpha},t)$ is the nodal state variable vector solution of Eq.(4), while the matrices $\boldsymbol{\Lambda}_{i,\mathbf{UU}}(\boldsymbol{\alpha},t)$ collect the *non-geometric spectral moments* (*NGSM*) [15-19]. After some algebra, the nodal *PEC* matrix, can be evaluated in time-domain, for quiescent structural systems (at time $t_0=0$), as follows:

$$\boldsymbol{\Sigma}_{\mathbf{ZZ}}(\boldsymbol{\alpha},t)=\int_{t_0}^{t}\int_{t_0}^{t}\boldsymbol{\Theta}\left(\boldsymbol{\alpha},t-\tau_1\right)\mathbf{w}\mathbf{w}^T\boldsymbol{\Theta}^T\left(\boldsymbol{\alpha},t-\tau_2\right)R_{FF}\left(\tau_1,\tau_2\right)\mathrm{d}\tau_1\,\mathrm{d}\tau_2 \tag{19}$$

where $\mathbf{w}$ is the vector defined in Eq.(5), $R_{FF}(\tau_1,\tau_2)$ is the complex autocorrelation function and $\boldsymbol{\Theta}(\boldsymbol{\alpha},t)$ is the transition matrix defined in Eq.(6). By substituting the transition matrix (6) into Eq.(19), the nodal *PEC* matrix can be written also as [20,21]:

$$\boldsymbol{\Sigma}_{\mathbf{ZZ}}(\boldsymbol{\alpha},t)=\boldsymbol{\Psi}^*(\boldsymbol{\alpha})\left\{\int_{t_0}^{t}\int_{t_0}^{t}\exp\left[\boldsymbol{\Lambda}^*\left(\boldsymbol{\alpha},t-\tau_1\right)\right]\mathbf{v}^*(\boldsymbol{\alpha})\mathbf{v}^T(\boldsymbol{\alpha})\exp\left[\boldsymbol{\Lambda}\left(\boldsymbol{\alpha},t-\tau_2\right)\right]R_{FF}\left(\tau_1,\tau_2\right)\mathrm{d}\tau_1\,\mathrm{d}\tau_2\right\}\boldsymbol{\Psi}^T(\boldsymbol{\alpha}) \tag{20}$$

where the vector $\mathbf{v}(\boldsymbol{\alpha})$ has been defined in Eq.(17). In this equation the autocorrelation function is defined as follows:

$$R_{FF}\left(t_1,t_2\right)=\int_0^\infty\exp\left[\mathrm{i}\omega\left(t_1-t_2\right)\right]a(\omega,t_1)\,a^*(\omega,t_2)G_0(\omega)\mathrm{d}\omega \tag{21}$$

where $a(\omega,t)\equiv a^*(-\omega,t)$ is the modulating function, that for *fully non-stationary processes* depends on both time and frequency. In Eq.(21) $G_0\left(\omega\right)$ is the one-sided *power spectral density* (*PSD*) function of the stationary counterpart of the fully not stationary input process having the one-sided *evolutionary PSD* (*EPSD*) defined as $G_{FF}(\omega,t)=\left|a(\omega,t)\right|^2G_0(\omega)$. By substituting Eq.(21) into Eq.(20), it is possible to evaluate the nodal *PEC* matrix (18) as:

$$\boldsymbol{\Sigma}_{\mathbf{ZZ}}(\boldsymbol{\alpha},t)=\int_0^\infty\mathbf{G}_{\mathbf{ZZ}}(\boldsymbol{\alpha},\omega,t)\,\mathrm{d}\,\omega=\boldsymbol{\Psi}^*(\boldsymbol{\alpha})\boldsymbol{\Sigma}_{\mathbf{XX}}(\boldsymbol{\alpha},t)\boldsymbol{\Psi}^T(\boldsymbol{\alpha}) \tag{22}$$

where $\boldsymbol{\Sigma}_{\mathbf{XX}}(\boldsymbol{\alpha},t)$ is the *PEC* matrix in the complex modal state subspace defined as:

$$\boldsymbol{\Sigma}_{\mathbf{XX}}(\boldsymbol{\alpha},t)=\int_0^\infty\mathbf{G}_{\mathbf{XX}}(\boldsymbol{\alpha},\omega,t)\,\mathrm{d}\,\omega \tag{23}$$

where $\mathbf{G}_{\mathbf{XX}}(\boldsymbol{\alpha},\omega,t)$ is the one-sided *EPSD* function matrix of the modal complex response, that is:

$$\mathbf{G}_{\mathbf{XX}}(\boldsymbol{\alpha},\omega,t)=G_0(\omega)\,\mathbf{X}^*(\boldsymbol{\alpha},\omega,t)\,\mathbf{X}^T(\boldsymbol{\alpha},\omega,t). \tag{24}$$

Notice that, in evaluating the nodal *PEC* matrix of Eq.(22), the following coordinate transformation has been introduced:

$$\mathbf{Z}(\boldsymbol{\alpha},\omega,t)=\boldsymbol{\Psi}(\boldsymbol{\alpha})\,\mathbf{X}(\boldsymbol{\alpha},\omega,t) \tag{25}$$

where $\mathbf{Z}(\boldsymbol{\alpha},\omega,t)$ is the *time-frequency varying response* (*TFR*) vector function of the nodal response, while $\mathbf{X}(\boldsymbol{\alpha},\omega,t)$ is the *TFR* vector function of the modal response, defined as:

$$\mathbf{X}(\boldsymbol{\alpha},\omega,t)=\int_{t_0}^{t}\exp\left[\boldsymbol{\Lambda}\left(\boldsymbol{\alpha},t-\tau\right)\right]\exp\left(\mathrm{i}\omega\tau\right)a(\omega,\tau)\mathrm{d}\tau\,\mathbf{v}(\boldsymbol{\alpha}). \tag{26}$$

In order to evaluate in explicit form the *TFR* vector function of modal response, the vector $\mathbf{X}(\boldsymbol{\alpha},\omega,t)$ can be evaluated as the solution of a set of $2m$ first order uncoupled differential equations, since the following relationship holds [20]:

$$\dot{\mathbf{X}}(\boldsymbol{\alpha},\omega,t)=\boldsymbol{\Lambda}(\boldsymbol{\alpha})\,\mathbf{X}(\boldsymbol{\alpha},\omega,t)+\mathbf{v}(\boldsymbol{\alpha})\exp\left(\mathrm{i}\omega t\right)a(\omega,t)\boldsymbol{\mathcal{U}}\left(t-t_0\right);\quad \mathbf{X}(\boldsymbol{\alpha},\omega,t_0)=\mathbf{X}_0(\boldsymbol{\alpha},\omega) \tag{27}$$

where $\mathbf{X}(\boldsymbol{\alpha},\omega,t_0)\equiv\mathbf{X}_0(\boldsymbol{\alpha},\omega)$ is the vector of the initial condition at time $t=t_0$ and $\mathcal{U}(t)$ is the *unit step function.*

If the particular solution of Eq.(27), $\mathbf{X}_{\mathrm{p}}(\boldsymbol{\alpha},\omega,t)$, can be determined in explicit form, the *TFR* vector function, according to the dynamics of non-classically damped systems, can be written as [21]:

$$\mathbf{X}(\boldsymbol{\alpha},\omega,t)=\left\{\mathbf{X}_{\mathrm{p}}(\boldsymbol{\alpha},\omega,t)+\exp\left[\boldsymbol{\Lambda}(\boldsymbol{\alpha})t\right]\left[\mathbf{X}_0(\boldsymbol{\alpha},\omega)-\mathbf{X}_{\mathrm{p}}(\boldsymbol{\alpha},\omega,t_0)\right]\right\}\mathcal{U}(t-t_0). \qquad (28)$$

The analytical expression of the particular solution vector $\mathbf{X}_{\mathrm{p}}(\boldsymbol{\alpha},\omega,t)$, can be easily obtained in closed form for the most common models of modulating function $a(\omega,t)$ proposed in literature [4-7]. It has been recently shown that the most useful time-frequency functions to model the fully non-stationary seismic excitation can be written as [6]:

$$a(\omega,t)=\varepsilon(\omega)\,(t-t_0)\,\exp\left[-\alpha_a(\omega)(t-t_0)\right]\mathcal{U}(t-t_0); \qquad (29)$$

where $\varepsilon(\omega)$ and $\alpha_a(\omega)$ could be complex functions which have to be chosen to satisfy the condition: $a(\omega,t)\equiv a^*(-\omega,t)$.

It has been demonstrated that for quiescent structural systems at time $t_0=0$, $\mathbf{X}_0(\boldsymbol{\alpha},\omega)=\mathbf{0}$, and for the modulating function, defined in Eq.(29), the vector $\mathbf{X}(\boldsymbol{\alpha},\omega,t)$, defined in Eq.(28), can be evaluated in explicit form as [20,21]:

$$\mathbf{X}(\boldsymbol{\alpha},\omega,t)=-\varepsilon(\omega)\left\{\exp\left[-\beta(\omega)\,t\right]\left[\boldsymbol{\Gamma}^2(\omega)+t\,\boldsymbol{\Gamma}(\omega)\right]-\exp\left[\boldsymbol{\Lambda}(\boldsymbol{\alpha})t\right]\boldsymbol{\Gamma}^2(\omega)\right\}\mathbf{v}(\boldsymbol{\alpha})\,\mathcal{U}(t) \qquad (30)$$

where $\beta(\omega)=\alpha_a(\omega)-\mathrm{i}\omega$ and $\boldsymbol{\Gamma}(\omega)$ is the diagonal matrix, function of the $\boldsymbol{\alpha}$ vector too that for simplicity‟s sake is omitted, defined as:

$$\boldsymbol{\Gamma}(\omega)\equiv\boldsymbol{\Gamma}(\boldsymbol{\alpha},\omega)=\left[\boldsymbol{\Lambda}(\boldsymbol{\alpha})+\beta(\omega)\mathbf{I}_{2m}\right]^{-1}. \qquad (31)$$

Then, it is possible to evaluate, in explicit form, the *EPSD* function matrix of the modal response by substituting Eq.(30) into Eq.(24) which can be written as:

$$\mathbf{G}_{\mathbf{ZZ}}(\boldsymbol{\alpha},\omega,t)=G_0(\omega)\,\boldsymbol{\Psi}^*(\boldsymbol{\alpha})\,\mathbf{X}^*(\boldsymbol{\alpha},\omega,t)\,\mathbf{X}^T(\boldsymbol{\alpha},\omega,t)\,\boldsymbol{\Psi}^T(\boldsymbol{\alpha}) \qquad (32)$$

### 3.2 Closed form solutions for the sensitivity of time-frequency varying response vector function

By differentiating the *PEC* matrix, defined in Eq.(18), it is possible to evaluate its sensitivity with respect to the *i*-th parameter, as follows:

$$\boldsymbol{\Sigma}_{\mathbf{s}_{\mathbf{Z},i}\mathbf{s}_{\mathbf{Z},i}}(\boldsymbol{\alpha}_0,t)=\left.\frac{\partial\boldsymbol{\Sigma}_{\mathbf{ZZ}}(\boldsymbol{\alpha},t)}{\partial\alpha_i}\right|_{\boldsymbol{\alpha}=\boldsymbol{\alpha}_0}=\mathrm{E}\left\langle\mathbf{Z}^*(\boldsymbol{\alpha}_0,t)\mathbf{s}_{\mathbf{Z},i}^T(\boldsymbol{\alpha}_0,t)\right\rangle+\mathrm{E}\left\langle\mathbf{Z}^*(\boldsymbol{\alpha}_0,t)\mathbf{s}_{\mathbf{Z},i}^T(\boldsymbol{\alpha}_0,t)\right\rangle^{*T} \qquad (33)$$

where the vector $\mathbf{s}_{\mathbf{Z},i}(\boldsymbol{\alpha}_0,t)$ has been defined in Eq.(12). It follows that, analogously to Eq.(22), the following relationship holds:

$$\mathrm{E}\left\langle\mathbf{Z}^*(\boldsymbol{\alpha}_0,t)\mathbf{s}_{\mathbf{Z},i}^T(\boldsymbol{\alpha}_0,t)\right\rangle=\boldsymbol{\Psi}^*(\boldsymbol{\alpha}_0)\left\{\int_0^\infty\mathbf{X}^*(\boldsymbol{\alpha}_0,\omega,t)\mathbf{Y}_i^T(\boldsymbol{\alpha}_0,\omega,t)\,G_0(\omega)\mathrm{d}\omega\right\}\boldsymbol{\Psi}^T(\boldsymbol{\alpha}_0) \qquad (34)$$

where the vector $\mathbf{Y}_i(\boldsymbol{\alpha}_0,\omega,t)$ is the *sensitivity* of *TFR* vector function with respect to the parameter $\alpha_i$:

$$\mathbf{Y}_i(\boldsymbol{\alpha}_0,\omega,t)=\int_0^t \exp\left[\boldsymbol{\Lambda}\left(\boldsymbol{\alpha}_0,t-\tau\right)\right]\mathbf{B}_i\left(\boldsymbol{\alpha}_0\right)\mathbf{X}(\boldsymbol{\alpha}_0,\omega,\tau)\,\mathrm{d}\tau. \tag{35}$$

Alternatively the sensitivity of *PEC* matrix can be defined as:

$$\boldsymbol{\Sigma}_{\mathbf{s}_{Z,i}\mathbf{s}_{Z,i}}\left(\boldsymbol{\alpha}_0,t\right)=\begin{bmatrix} \left.\dfrac{\partial}{\partial\alpha_i}\boldsymbol{\Lambda}_{0,\mathbf{UU}}(\boldsymbol{\alpha},t)\right|_{\boldsymbol{\alpha}=\boldsymbol{\alpha}_0} & \left.\mathrm{i}\dfrac{\partial}{\partial\alpha_i}\boldsymbol{\Lambda}_{1,\mathbf{UU}}(\boldsymbol{\alpha},t)\right|_{\boldsymbol{\alpha}=\boldsymbol{\alpha}_0} \\ \left.-\mathrm{i}\dfrac{\partial}{\partial\alpha_i}\boldsymbol{\Lambda}^{*T}_{1,\mathbf{UU}}(\boldsymbol{\alpha},t)\right|_{\boldsymbol{\alpha}=\boldsymbol{\alpha}_0} & \left.\dfrac{\partial}{\partial\alpha_i}\boldsymbol{\Lambda}_{2,\mathbf{UU}}(\boldsymbol{\alpha},t)\right|_{\boldsymbol{\alpha}=\boldsymbol{\alpha}_0}\end{bmatrix}=\int_0^\infty \mathbf{G}_{\mathbf{s}_{Z,i}\mathbf{s}_{Z,i}}\left(\boldsymbol{\alpha}_0,t\right)\mathrm{d}\,\omega \tag{36}$$

whose elements are the sensitivity of first three spectral moments with respect to the parameter $\alpha_i$. In this equation $\mathbf{G}_{\mathbf{s}_{Z,i}\mathbf{s}_{Z,i}}\left(\boldsymbol{\alpha}_0,t\right)$ is the sensitivity of the one-sided *EPSD* function of nodal response, that is:

$$\begin{aligned}\mathbf{G}_{\mathbf{s}_{Z,i}\mathbf{s}_{Z,i}}\left(\boldsymbol{\alpha}_0,t\right)&=\left.\frac{\partial\mathbf{G}_{\mathbf{ZZ}}(\boldsymbol{\alpha},\omega,t)}{\partial\alpha_i}\right|_{\boldsymbol{\alpha}=\boldsymbol{\alpha}_0}\\&=G_0(\omega)\,\boldsymbol{\Psi}^*(\boldsymbol{\alpha})\left[\mathbf{X}^*(\boldsymbol{\alpha}_0,\omega,t)\,\mathbf{Y}_i^T(\boldsymbol{\alpha}_0,\omega,t)+\mathbf{Y}_i^*(\boldsymbol{\alpha}_0,\omega,t)\,\mathbf{X}^T(\boldsymbol{\alpha}_0,\omega,t)\right]\boldsymbol{\Psi}^T(\boldsymbol{\alpha}).\end{aligned} \tag{37}$$

The main problem is now to evaluate the vector $\mathbf{Y}_i(\boldsymbol{\alpha}_0,\omega,t)$, defined in Eq.(35), taking into account Eq.(30). This vector function can be evaluated as solution of the following differential equation with zero start conditions at time $t_0=0$:

$$\dot{\mathbf{Y}}_i(\boldsymbol{\alpha}_0,\omega,t)=\boldsymbol{\Lambda}(\boldsymbol{\alpha}_0)\,\mathbf{Y}_i(\boldsymbol{\alpha}_0,\omega,t)+\mathbf{B}_i\left(\boldsymbol{\alpha}_0\right)\mathbf{X}(\boldsymbol{\alpha}_0,\omega,\tau)\,\mathcal{U}\left(t-t_0\right);\quad \mathbf{Y}_i(\boldsymbol{\alpha}_0,\omega,0)=\mathbf{0}. \tag{38}$$

To perform the solution of this set of differential equations the vector defined in Eq.(30) is rewritten as:

$$\mathbf{X}\left(\boldsymbol{\alpha}_0,\omega,t\right)=\mathbf{X}_1\left(\boldsymbol{\alpha}_0,\omega,t\right)+\mathbf{X}_2\left(\boldsymbol{\alpha}_0,\omega,t\right) \tag{39}$$

where

$$\begin{aligned}\mathbf{X}_1\left(\boldsymbol{\alpha}_0,\omega,t\right)&=-\varepsilon(\omega)\exp\left[-\beta(\omega)\,t\right]\left[\boldsymbol{\Gamma}^2(\omega)+t\,\boldsymbol{\Gamma}(\omega)\right]\mathbf{v}(\boldsymbol{\alpha}_0)\,\mathcal{U}\left(t\right);\\ \mathbf{X}_2\left(\boldsymbol{\alpha}_0,\omega,t\right)&=\varepsilon(\omega)\exp\left[\boldsymbol{\Lambda}\left(\boldsymbol{\alpha}_0\right)t\right]\boldsymbol{\Gamma}^2(\omega)\mathbf{v}(\boldsymbol{\alpha}_0)\,\mathcal{U}\left(t\right).\end{aligned} \tag{40}$$

It follows that it is possible to split the vector solution of Eq.(38) as the sum of two vectors, solutions of the following two sets of differential equations, with zero start initial conditions at time $t_0=0$:

$$\begin{aligned}\dot{\mathbf{Y}}_{i,1}(\boldsymbol{\alpha}_0,\omega,t)&=\boldsymbol{\Lambda}(\boldsymbol{\alpha}_0)\,\mathbf{Y}_{i,1}(\boldsymbol{\alpha}_0,\omega,t)+\mathbf{B}_i\left(\boldsymbol{\alpha}_0\right)\mathbf{X}_1(\boldsymbol{\alpha}_0,\omega,t);\quad \mathbf{Y}_{i,1}(\boldsymbol{\alpha}_0,\omega,0)=\mathbf{0}\\ \dot{\mathbf{Y}}_{i,2}(\boldsymbol{\alpha}_0,\omega,t)&=\boldsymbol{\Lambda}(\boldsymbol{\alpha}_0)\,\mathbf{Y}_{i,2}(\boldsymbol{\alpha}_0,\omega,t)+\mathbf{B}_i\left(\boldsymbol{\alpha}_0\right)\mathbf{X}_2(\boldsymbol{\alpha}_0,\omega,t);\quad \mathbf{Y}_{i,2}(\boldsymbol{\alpha}_0,\omega,0)=\mathbf{0}\end{aligned} \tag{41}$$

It follows that the *sensitivity TFR„s* vector function can be evaluate in closed form solution as:

$$\begin{aligned}\mathbf{Y}_i\left(\boldsymbol{\alpha}_0,\omega,t\right)=\mathbf{Y}_{i,1}\left(\boldsymbol{\alpha}_0,\omega,t\right)+\mathbf{Y}_{i,2}\left(\boldsymbol{\alpha}_0,\omega,t\right)=\Big\{&\mathbf{Y}_{i,1,\mathrm{p}}\left(\boldsymbol{\alpha}_0,\omega,t\right)+\mathbf{Y}_{i,2,\mathrm{p}}\left(\boldsymbol{\alpha}_0,\omega,t\right)\\&-\exp\left[\boldsymbol{\Lambda}\left(\boldsymbol{\alpha}_0\right)t\right]\left[\mathbf{Y}_{i,1,\mathrm{p}}\left(\boldsymbol{\alpha}_0,\omega,0\right)+\mathbf{Y}_{i,2,\mathrm{p}}\left(\boldsymbol{\alpha}_0,\omega,0\right)\right]\Big\}\,\mathcal{U}\left(t\right)\end{aligned} \tag{42}$$

where the particular solution vectors of Eqs.(41), can be evaluated, after some algebra, as follows:

$$\mathbf{Y}_{i,1,\mathrm{p}}\left(\boldsymbol{\alpha}_0,\omega,t\right)=\varepsilon(\omega)\exp\left[-\beta(\omega)\,t\right]\boldsymbol{\Gamma}(\omega)\left[\boldsymbol{\Gamma}(\omega)\mathbf{B}_i\left(\boldsymbol{\alpha}_0\right)+\mathbf{B}_i\left(\boldsymbol{\alpha}_0\right)\boldsymbol{\Gamma}(\omega)+t\,\mathbf{B}_i\left(\boldsymbol{\alpha}_0\right)\right]\boldsymbol{\Gamma}(\omega)\mathbf{v}(\boldsymbol{\alpha}_0);$$
$$\mathbf{Y}_{i,2,\mathrm{p}}(\boldsymbol{\alpha}_0,\omega,t)=\varepsilon(\omega)\mathbf{P}_i(\boldsymbol{\alpha}_0,t)\exp\left[\boldsymbol{\Lambda}\left(\boldsymbol{\alpha}_0\right)t\right]\boldsymbol{\Gamma}^2(\omega)\mathbf{v}(\boldsymbol{\alpha}_0);\tag{43}$$

where $\mathbf{P}_i(\boldsymbol{\alpha}_0,t)$ is a matrix of order $(2m\times 2m)$ whose elements, $P_{i,jk}(\boldsymbol{\alpha}_0,t)$, are defined as follows:

$$P_{i,jj}(\boldsymbol{\alpha}_0,t)=t\,B_{i,jj}(\boldsymbol{\alpha}_0);\quad P_{i,jk}(\boldsymbol{\alpha}_0,t)=\frac{B_{i,jk}(\boldsymbol{\alpha}_0)}{\lambda_k-\lambda_j},\,j\neq k\tag{44}$$

with $B_{i,jk}(\boldsymbol{\alpha}_0)$ elements of the matrix $\mathbf{B}_i\left(\boldsymbol{\alpha}_0\right)$. Finally, the sensitivity of *PEC* matrix with respect to the parameter $\alpha_i$, defined in Eq.(36), can be evaluated by substituting Eqs.(39) and (42) into Eq.(37), and then the result (the explicit closed form of the nodal *EPSD* function matrix) into Eq.(36).

## 4 NUMERICAL APPLICATIONS

In this section, the accuracy of the proposed procedure has been verified, through the comparison of the results of a numerical application with the *Monte Carlo Simulation* (*MCS*) method (1000 samples). The analysed system is composed by two interconnected three-story selected structures, having the same floor elevation, as depicted in Figure 1. The two neighbouring floors are connected by a damper device. Each fluid damper device is modelled as a combination of a linear spring, having stiffness $k_{\mathrm{d},i}=\left(1+\alpha\right)\times 10^5\,\mathrm{N/m}$, and a linear dashpot, having damping coefficient $c_{\mathrm{d},i}=\left(1+\alpha\right)\times 10^6\,\mathrm{N\,s/m}$, with $\alpha>0$ a dimensionless parameter. It follows that the vector $\boldsymbol{\alpha}$ becomes a scalar quantity and the nominal structural matrices are evaluated setting $\alpha=0$.

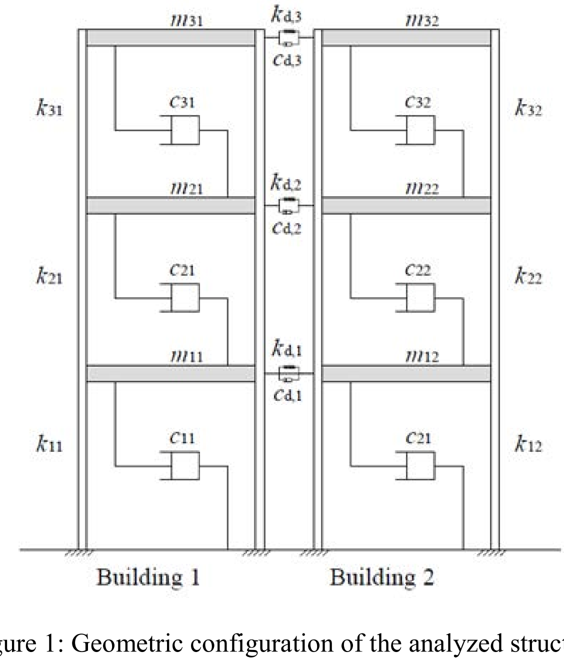


Figure 1: Geometric configuration of the analyzed structure.

The characteristics of each floor (mass $m_i$, stiffness $k_i$ and damping coefficient $c_i$) for the two buildings are summarized in Table 1. In Table 2 the modal characteristics of the two un-linked buildings (circular frequency $\omega_i$ , period $T_i$ and modal participating mass ratio $\varepsilon_i$ ), together with the global system are reported.

| | Building 1 | Building 2 |
|---|---|---|
| $k_i[\mathrm{N/m}]$ | $2\times10^{11}$ | $2\times10^{9}$ |
| $m_i[\mathrm{kg}]$ | $1.29\times10^{6}$ | $1.29\times10^{6}$ |
| $c_i[\mathrm{Ns/m}]$ | $1\times10^{5}$ | $1\times10^{5}$ |

Table 1: Characteristics of the analysed buildings.

| Building 1 | | | Building 2 | | | Global system | | |
|---|---|---|---|---|---|---|---|---|
| $\omega_i[\mathrm{rad/s}]$ | $T_i[\mathrm{s}]$ | $\varepsilon_i[\%]$ | $\omega_i[\mathrm{rad/s}]$ | $T_i[\mathrm{s}]$ | $\varepsilon_i[\%]$ | $\omega_i[\mathrm{rad/s}]$ | $T_i[\mathrm{s}]$ | $\varepsilon_i[\%]$ |
| 175.235 | 0.036 | 91.408 | 17.523 | 0.359 | 91.408 | 17.527 | 0.359 | 47.704 |
| 490.998 | 0.013 | 7.488 | 49.099 | 0.128 | 7.488 | 49.100 | 0.128 | 3.744 |
| 709.512 | 0.009 | 1.104 | 70.951 | 0.089 | 1.104 | 70.952 | 0.089 | 0.552 |

Table 2: Modal information of the analysed buildings.

The selected structures are subjected to a fully non-stationary seismic input whose *EPSD* function can be expressed as:

$$G_{FF}(\omega,t) = a^2(\omega,t)\, G_0(\omega). \tag{45}$$

In the previous equation the parameters of the modulating function, defined in Eq.(29), have been set as: $\alpha_a(\omega) = \frac{1}{2}\left(0.15 + \frac{\omega^2}{25\pi^2}\right)$, $\varepsilon(\omega) = \frac{\sqrt{2}}{5\pi}\omega$ and $t_0 = 0$; the Tajimi-Kanai *PSD* function is used to model the *PSD* function of the stationary counterpart of the input stochastic process:

$$G_0(\omega) = G_{\mathrm{W}} \frac{4\zeta_{\mathrm{K}}^2\, \omega_{\mathrm{K}}^2\, \omega^2 + \omega_{\mathrm{K}}^4}{\left(\omega_{\mathrm{K}}^2 - \omega^2\right)^2 + 4\zeta_{\mathrm{K}}^2\, \omega_{\mathrm{K}}^2\, \omega^2} \tag{46}$$

where $G_{\mathrm{W}} = 0.05\ \mathrm{m^2/s^3}$ , $\omega_{\mathrm{K}} = 4\,\pi$ rad/s is the filter frequency that determines the dominant input frequency and $\zeta_{\mathrm{K}} = 0.6$ is the filter damping coefficient that indicates the sharpness of the *PSD* function.

In order to show the accuracy of the proposed method, the *sensitivity* $S_{\lambda_{\ell,u_r}}(t)$ (indicated in Eq.(47)) with respect to the parameter $\alpha$ of *NGSM*s of the generic $r$-th floor displacement $u_r(t)$ are compared with *MCS* results and are depicted in Figures 2-4.

$$S_{\lambda_{\ell,u_r}}(t) = \left.\frac{\partial \lambda_{\ell,u_r}(\alpha,t)}{\partial \alpha}\right|_{\alpha=0} \tag{47}$$

In Eq.(47), $\lambda_{\ell,u_r}(\alpha,t)$ are the $r$-diagonal elements of the $\mathbf{\Lambda}_{\ell,\mathbf{UU}}(\boldsymbol{\alpha},t)$ matrix defined in Eq.(36) while the subscript $\ell$ indicates the order of the *NGSM*s.

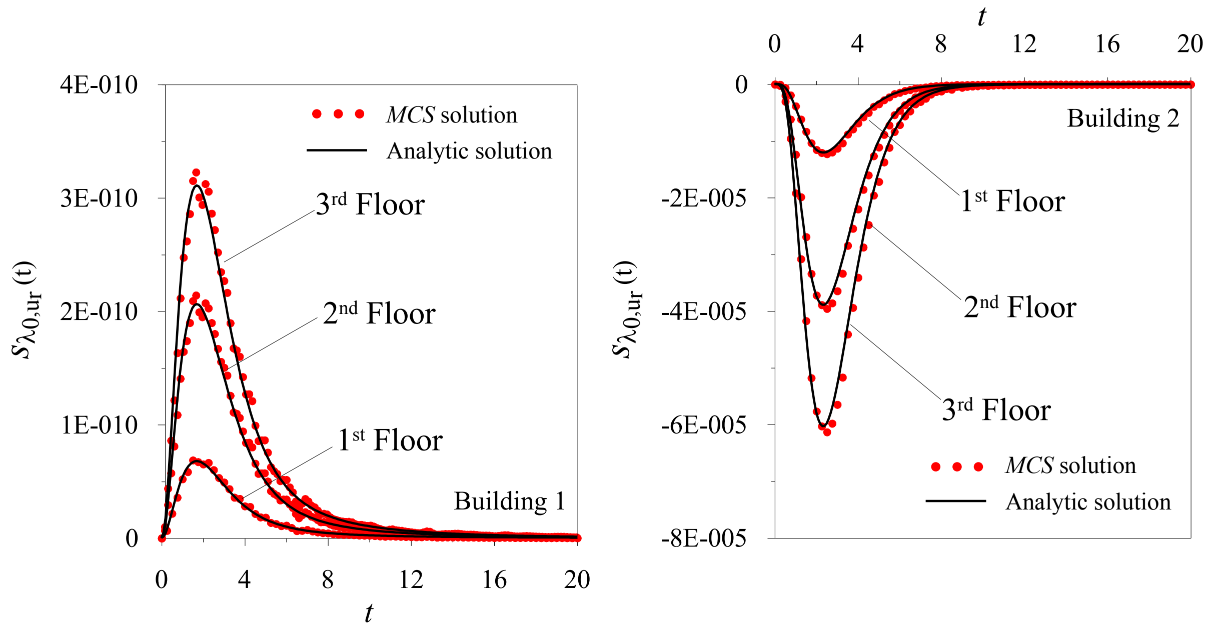


Figure 2: Time histories of the sensitivity of the *NGSM* $\lambda_{0,u_r}(t)$, for the six relative to ground floor displacements of the buildings (black line) and comparison with the *MCS* (red dots).

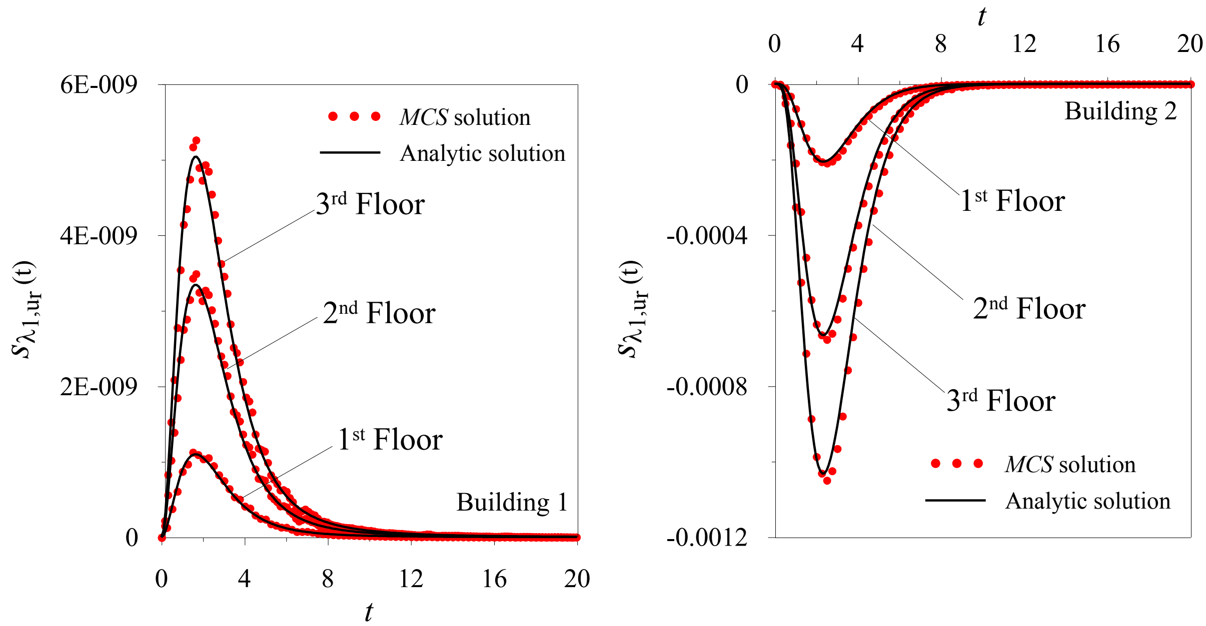


Figure 3: Time histories of the sensitivity of real part of the *NGSM* $\lambda_{1,u_r}(t)$, for the six relative to ground floor displacements of the buildings (black line) and comparison with the *MCS* (red dots).

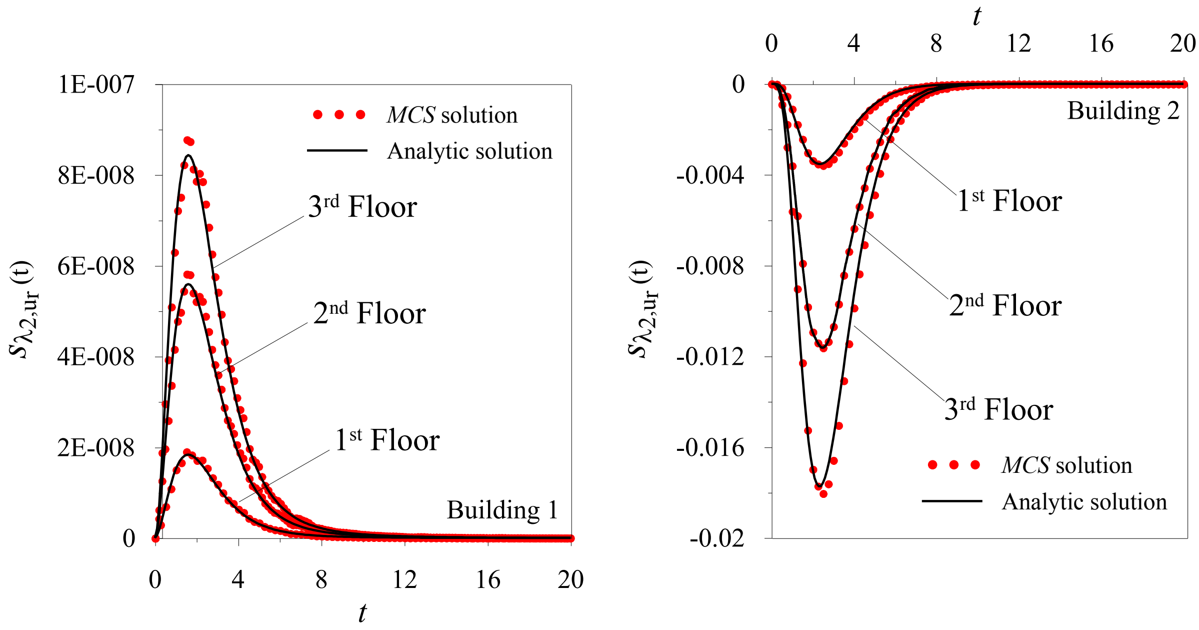


Figure 4: Time histories of the sensitivity of the *NGSM* $\lambda_{2,u_r}(t)$, for the six relative to ground floor displacements of the buildings (black line) and comparison with the *MCS* (red dots).

The figures 2-4 evidence a perfect superposition between the proposed analytical solution and the *MCS* method, demonstrating the accuracy of the proposed procedure. Obviously a positive sensitivity indicates an increment of the corresponding *NGSM*, when the parameter $\alpha$ changes, while a negative sensitivity means that the *NGSM* decreases when the parameter changes.

Finally, in Figure 5 the *sensitivity* of the first *NGSM* of the response of third floors, for both buildings, and for five different ratio of the stiffness, have been depicted. In the first three cases the stiffness of building 1 is assumed: $k_i = 2\times10^{11}\,\text{N/m}$. In the latest two cases the stiffness of building 2 is: $k_i = 2\times10^{11}\,\text{N/m}$. These figures show that the sensitivity of the first *NGSM* is positive for the more rigid building, while it is negative for the more deformable building. Namely, for the presence of devices, changing the parameter $\alpha$ the response of more rigid structures increases, while the response of lighter structures decreases. In the third case, when the two structures have the same stiffness, the sensitivity for building 1 is negative for the former time instants and positive for the following ones. The opposite result is obtained for building 2. This means that in the third case the sign of the sensitivity changes in the time. Zero sensitivity means that a change of parameter $\alpha$ does not modify the response of two buildings with respect to the nominal case.

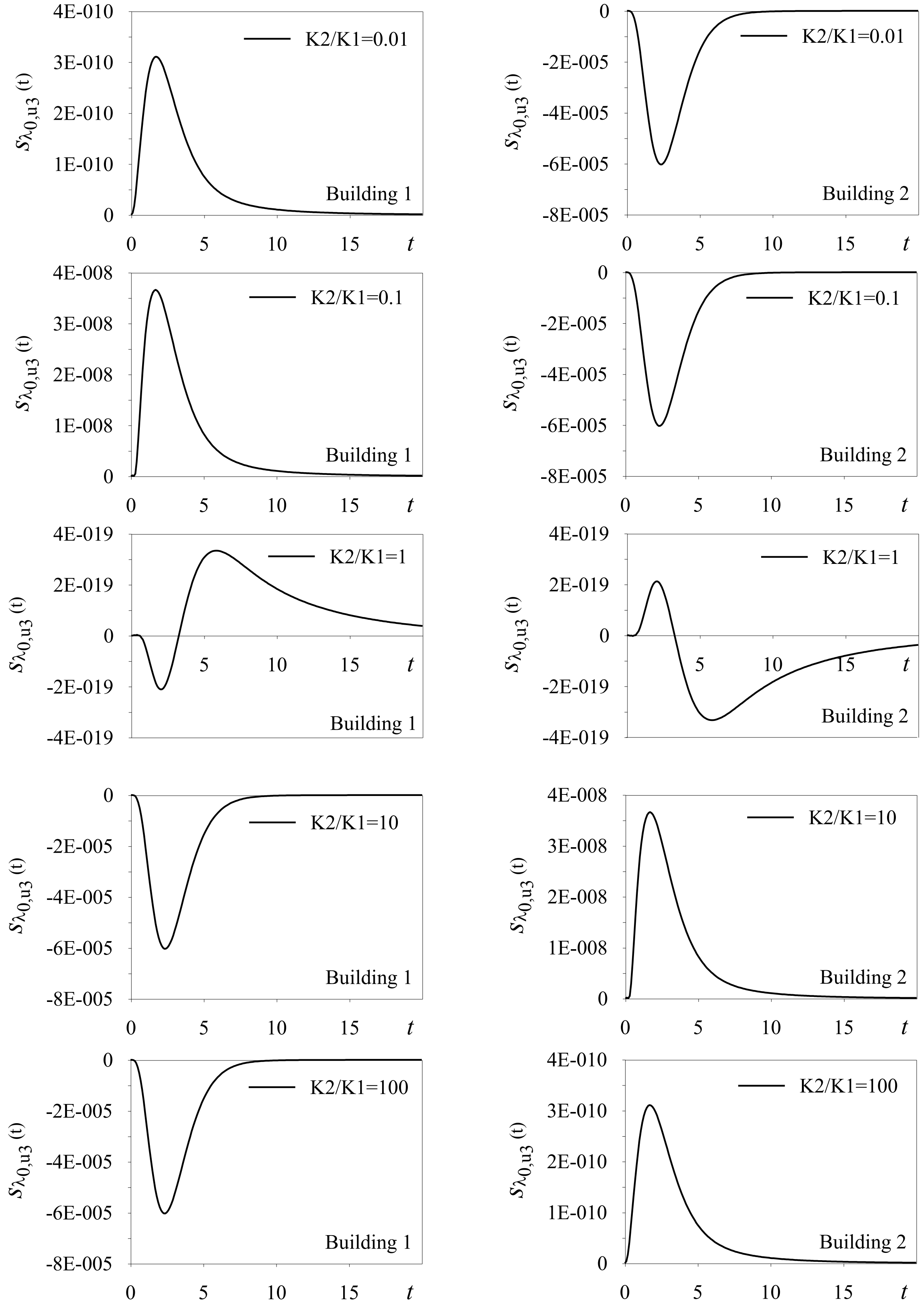


Figure 5: Time histories of the sensitivity of the *NGSM* $\lambda_{0,u_3}(t)$, for the third relative to ground floor displacements of the buildings for different ratio of the stiffness.

## 5 CONCLUSIONS

In the framework of optimization procedures, especially during the design of vibration control devices, the sensitivity analysis is a very powerful tool to evaluate how the structural response is modified with reference structural parameters changes.

In this paper a novel method for the evaluation of the sensitivities of *non-geometric spectral moments* of the structural response of linear classically or non-classically damped linear structural systems subjected to both separable and non-separable non-stationary excitations is proposed.

The proposed procedure is based on two fundamental steps: first, it is necessary to determine sensitivities of *evolutionary frequency response functions*, and it is possible thanks to the herein obtained explicit closed form solutions; then, by simple frequency domain integrals, it is possible to evaluate the sensitivity of the structural response statistics.

The presented method has a unified approach for both classically and non-classically damped discrete linear structural systems, thanks to use of the state-variables.

The numerical application on a plane-frame demonstrated the effectiveness of the proposed method, since a validation with *MCS* method has been done.

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