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# DATA FEATURES-BASED LIKELIHOOD-INFORMED BAYESIAN FINITE ELEMENT MODEL UPDATING

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## Abstract

*A new formulation for likelihood-informed Bayesian inference is proposed in this work based on probability models introduced for the features between the measurements and model predictions. The formulation applies to both linear and nonlinear dynamic models of structures. A relation between likelihood-informed and likelihood-free approximate Bayesian computation (ABC) is also established in this study, demonstrating that both formulations yield reasonable and consistent uncertainties for the model parameters. In particular, the uncertainties obtained with the new formulation account better for the fact that different sampling rates used in recording response time history measurements often yield measurements that contain the same information and so the sampling rate should not affect the uncertainty in the model parameters. The effectiveness of the proposed approach is demonstrated using an example from model updating of a linear model of a dynamical spring-mass chain system.*

**Keywords:** Uncertainty quantification, Bayesian learning, model updating, structural dynamics, Likelihood-informed Bayesian computation, data features


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# 1 INTRODUCTION

Bayesian model updating has gained more interest because of its effectiveness in practical engineering problems [1-3]. In Bayesian updating, the prior probability density function (PDF) of model parameters is updated to the posterior PDF by accounting for the information obtained from the measurements. Using probability models for the prediction errors, often formulated as the discrepancy between model predictions and the measurements, the likelihood function is developed. Asymptotic methods and sampling techniques have been developed to solve the parameter inference problem. In particular, sampling methods include versions of Markov Chain Monte Carlo (MCMC) (e.g. [4]), adaptive MCMC [5] as well as Transitional MCMC (TMCMC) [6, 7]. For likelihood-free parameter inference, the approximate Bayesian computation (ABC) has been developed. Among the algorithms proposed to solve the ABC, the subset simulation [8, 9] is shown to be computational effective alternative.

Bayesian model updating in structural dynamics using response time histories measurements such as accelerations, displacements or strains is often formulated by introducing point-to-point probabilistic descriptions of the discrepancy between the measurements and model predictions [10]. Spatially and temporally uncorrelated prediction error models used to quantify these discrepancies, result in very peaked posterior probability distributions for the model parameters due to the large number of data points available from high sampling rates. Spatially and temporally correlated prediction error models are more reasonable for quantifying uncertainties [11, 12]. However, the uncertainty depends on the correlation structure assumed which is often unknown and needs to be selected from a family of user-introduced correlation structures that might not be representative for the application. In general, the uncertainty quantified by the posterior probability distribution depends highly on the prediction error models and the correlation structure introduced between time instances as well as between measurements at different locations.

Herein we address the problem of Bayesian learning given response time history measurements. It is expected that for sufficiently small sampling rate, the information contained in the response time histories is independent of the sampling rate used to represent the time histories. Conventional techniques fail to quantify such independence and also give unrealistically small uncertainties due to the large number of data points used to represent the time histories. To properly quantify uncertainties, we propose a new formulation for likelihood-informed Bayesian inference based on probability models introduced for the features between the measured data and model predictions. Specifically, a probability model is assigned to the square of the discrepancy of the response time history between the measurement and the model prediction. Different probability models are investigated, such as a truncated Gaussian model and an exponential distribution model. It is demonstrated that reasonable uncertainties are obtained for the model parameters that are independent of the sampling rate used to represent the response time histories. A relation between likelihood-informed and likelihood-free Bayesian computations is also established, demonstrating that both formulations yield reasonable and consistent uncertainties for the model parameters. A spring-mass chain model with simulated, noise contaminated, measured acceleration time histories is used to demonstrate the effectiveness of the proposed approach.

The rest of this paper is organized as follows. In Section 2, the new likelihood-informed formulation for Bayesian model updating is proposed and compared with ABC formulation. The effectiveness of the proposed method is demonstrated using a spring-mass chain system in Section 3. Section 4 reports the conclusions of this study.

## 2 PROPOSED BAYESIAN FORMULATIONS

In Bayesian framework, the probabilities of unknown parameter sets $\underline{\theta}$ in the model class $M$ can be first estimated from the prior probability density functions (PDF), and then it can be updated based on the following Bayesian formula when some measurements $D$ are available:

$$p(\underline{\theta} \mid D, M) = \mathrm{c}\, p(D \mid \underline{\theta}, M)\, p(\underline{\theta} \mid M) \tag{1}$$

where $p(\underline{\theta} \mid D, M)$ is the posterior PDF of the model parameters given the measurements $D$ and the model class $M$ ; $p(\underline{\theta} \mid M)$ is the prior PDF; c is the constant which is selected so that the posterior PDF integrates to one; $p(D \mid \underline{\theta}, M)$ is the likelihood function of observing the data from the model class.

### 2.1 Model parameter estimation

Consider a parameterized class of structures models $\underline{g}(\underline{\theta}; M)$, where $M$ is the model, $\underline{\theta}$ is the set of model parameters which can be identified using the measurements $D$ . Let $D = \left\{ \hat{y}_j(k\Delta t) \in R^{N_0}, j = 1, 2, \cdots, N_0; k = 1, 2, \cdots, N_D \right\}$ be the measured response time histories data from the structure, where $N_0$ is the number of degrees of freedom (DOF) of the models, $N_D$ is the number of the sampled data using a sampling rate $\Delta t$ , $j$ and $k$ denote the $j$-th modes and time index at time $k\Delta t$ , respectively.

Conventional methods for parameter estimation in structural dynamics using direct response time history measurements are based on prediction error equations formulated at time $t = k\Delta t$ as follows:

$$\hat{y}_j(k) = g_j(k; \underline{\theta}, M) + \varepsilon_j(k; \underline{\theta}) \tag{2}$$

Using a zero-mean Gaussian model for the prediction errors $\varepsilon_j(k; \underline{\theta})$, $k = 1, 2, \cdots, N_D$ , and assuming of the prediction errors between the different sensor DOF $j = 1, 2, \cdots, N_0$ , one can readily built the likelihood in the form given in [10, 13].

Herein, a new formulation for the likelihood is presented based on introducing probabilistic models for the features between the data and the model predictions. Specifically, it is assumed that the average of the square of the discrepancy between the measurements $\hat{y}_j(k)$ and the model prediction $g_j(k; \underline{\theta} \mid M)$, $k = 1, 2, \cdots, N_D$ , satisfy the following equation:

$$\frac{1}{N_D} \sum_{k=1}^{N_D} \left[ \hat{y}_j(k) - g_j(k; \underline{\theta}; M) \right]^2 = e_j \tag{3}$$

Due to the fact that the square error is always larger than zero, the uncertainty in $e_j$ can be quantified with the following two kinds of distributions: 1. the truncated normal distribution; 2. the exponential distribution.

Regarding the case 1, the PDF of each variable $e_j$ can be written as [14]:

$$p(e_j) = \frac{\sqrt{2}}{\sqrt{\pi}\sigma} \exp\left(-\frac{e_j^2}{2\sigma^2}\right) \tag{4}$$

where $\sigma$ is the prediction error parameter of the truncated Gaussian probability models. The likelihood-informed based on the data features can be derived by the following formula:

$$p(\underline{e}|\ \underline{\theta},M)=\prod_{j=1}^{N_0} p(e_j|\underline{\theta},M) \tag{5}$$

The proposed likelihood is then given by:

$$p(\underline{e}|\ \underline{\theta},M)=\left(\frac{\sqrt{2}}{\sqrt{\pi}}\right)^{N_0}\cdot\left(\frac{1}{\sigma}\right)^{N_0}\cdot\exp\left(-\frac{1}{2}J\left(\underline{\theta};\sigma;M\right)\right) \tag{6}$$

where $J\left(\underline{\theta};\sigma;M\right)=\frac{1}{\sigma^2}\sum_{j=1}^{N_0}J_j\left(\underline{\theta};M\right)$, and $J_j\left(\underline{\theta};M\right)=\frac{1}{N_D}\sum_{k=1}^{N_D}\left[\hat{y}_j\left(k\right)-g_j\left(k;\underline{\theta};M\right)\right]^2$. Consequently, the logarithmic of the likelihood is:

$$L\left(\underline{\theta}\right)=\ln p(\underline{e}|\ \underline{\theta},M)=\mathrm{c}_0-N_0\ln\sigma-\frac{1}{2\sigma^2 N_D}\sum_{j=1}^{N_0}\sum_{k=1}^{N_D}\left[\hat{y}_j\left(k\right)-g_j\left(k;\underline{\theta};M\right)\right]^2 \tag{7}$$

where $c_0=N_0\ln\sqrt{\frac{2}{\pi}}$.

In the case 2, the PDF of each variable $e_j$, assuming that it follows an exponential distribution, is given by:

$$p(e_j)=\begin{cases}\lambda\exp(-\lambda e_j) & e_j\geq 0\\ 0 & e_j<0\end{cases} \tag{8}$$

where the parameter $\lambda$ is reparameterized by $\lambda=\frac{1}{2\sigma^2}$, which can make the exponent term equal to that of the truncated normal distribution. Similarly, the logarithmic likelihood function is calculated as:

$$L\left(\underline{\theta}\right)=\ln p(\underline{e}|\ \underline{\theta},M)=\mathrm{c}_1-2N_0\ln\sigma-\frac{1}{2\sigma^2 N_D}\sum_{j=1}^{N_0}\sum_{k=1}^{N_D}\left[\hat{y}_j\left(k\right)-g_j\left(k;\underline{\theta};M\right)\right]^2 \tag{9}$$

where $c_1=-N_0\ln 2$.

When the prior PDF and likelihood function are determined, the posterior PDF of the model parameters $\underline{\theta}$ is further solved according to the Eq. (1). It should be noted that the new method extends a recent likelihood-informed formulation developed for the case where the modal frequencies and mode shape components are available as the measured data [15].

Several methods have been introduced to estimate the model parameters and their uncertainties. Specifically, Monte Carlo Markov Chain (MCMC) [4], adaptive MCMC [5] as well as Transitional MCMC (TMCMC) [6], etc, can be used for populating with samples the support of the posterior distribution. Herein, the TMCMC algorithm is applied.

### 2.2 Relationship between likelihood-informed formulation and ABC

Based on the Eq. (9), the most probable value (MPV) $\hat{\sigma}^2$ of the posterior PDF can be obtained. Equivalently, it can be solved by maximizing the logarithmic likelihood function $L$:

$$\left.\frac{\partial L_1}{\partial \sigma^2}\right|_{\sigma^2=\hat{\sigma}^2} = 0 \tag{10}$$

where $L_1 = -L$. The best estimate is then given by:

$$\hat{\sigma}^2 = \frac{1}{2N_0}\varepsilon \tag{11}$$

where $\varepsilon$ is defined as a prediction error, which is given by:

$$\varepsilon = \sum_{j=1}^{N_0} \frac{1}{N_D} \sum_{k=1}^{N_D} \left[\hat{y}_j(k) - g_j(k;\underline{\theta};M)\right]^2 \tag{12}$$

Equivalently, Eq. (11) can be rewritten as follows:

$$\varepsilon = 2N_0\hat{\sigma}^2 \tag{13}$$

In ABC algorithms, a summary statistics $\eta$ and a tolerance parameter $\delta$ are first introduced [16]:

$$\rho\big(\eta(X), \eta(D)\big) \le \delta \tag{14}$$

where $X \in D$ denotes a simulated dataset from $p(D|\underline{\theta}, M)$, and $\rho(\cdot,\cdot)$ is a distance measure on the model output space. In general, the measure $\rho(\cdot,\cdot)$ is chosen to be the least square measure of the distance between the measurements and the model prediction from a parameterized class of structures models. Specifically for the model with predictions $\underline{g}(\underline{\theta};M)$, it is written as:

$$\rho = \sum_{j=1}^{N_0} \frac{1}{N_D} \sum_{k=1}^{N_D} \left[\hat{y}_j(k) - g_j(k;\underline{\theta};M)\right]^2 \tag{15}$$

It can be readily found that the right side term in Eq. (15) is exactly the same as that in Eq. (12), thus the tolerance value $\delta$ can be then selected based on the best estimate $\hat{\sigma}^2$:

$$\delta = 2N_0\hat{\sigma}^2 \tag{16}$$

The effectiveness of choosing the tolerance value is also demonstrated using examples in the next section.

## 3 NUMERICAL EXAMPLE

### 3.1 Description of a 10-DOF Spring-Mass Chain model

Consider a 10-DOF spring-mass chain system excited at the base. The equation of motion with base excitation $\ddot{y}_g(t)$:

$$M\underline{\ddot{y}}(t) + C\underline{\dot{y}}(t) + K(\underline{\theta})\underline{y}(t) = -M\underline{1}\ddot{y}_g(t) \tag{17}$$

where $\underline{1} = [1, 1, \cdots, 1]^T$ is a $10\times 1$ vector. The system is created based on the following assumptions:

a) The mass matrix $M$ is diagonal having elements equal to 1kg.

b) The springs are assumed to have the same stiffness equal to 1000N/m, and the spring matrix $K$ is given by the following stiffness matrix when the parameter $\underline{\theta}=\underline{1}$:

$$K=\begin{bmatrix} k_1+k_2 & -k_2 & 0 & \cdots & 0 \\ -k_2 & k_2+k_3 & -k_3 & 0 & 0 \\ 0 & -k_3 & \ddots & \ddots & 0 \\ 0 & 0 & \ddots & k_9+k_{10} & -k_{10} \\ 0 & 0 & 0 & -k_{10} & k_{10} \end{bmatrix} \tag{18}$$

c) Rayleigh damping is assumed with the damping matrix written as

$$C=\alpha M+\beta K \tag{19}$$

where the coefficient $\alpha$ and $\beta$ are taken to be 0.2265 and $6.7515e^{-4}$, respectively, corresponding to given damping ratios $\zeta_1=\zeta_5=0.02$ for the first and fifth modes of the system. .

d) Given the above system properties, the natural frequencies $\omega_1,\omega_2,\omega_3$ of the first three modes are estimated to be 1.0Hz, 3.0Hz and 4.9Hz.

e) The base excitation $\ddot{y}_g$ is obtained from an earthquake excitation, as shown in Fig. 1.

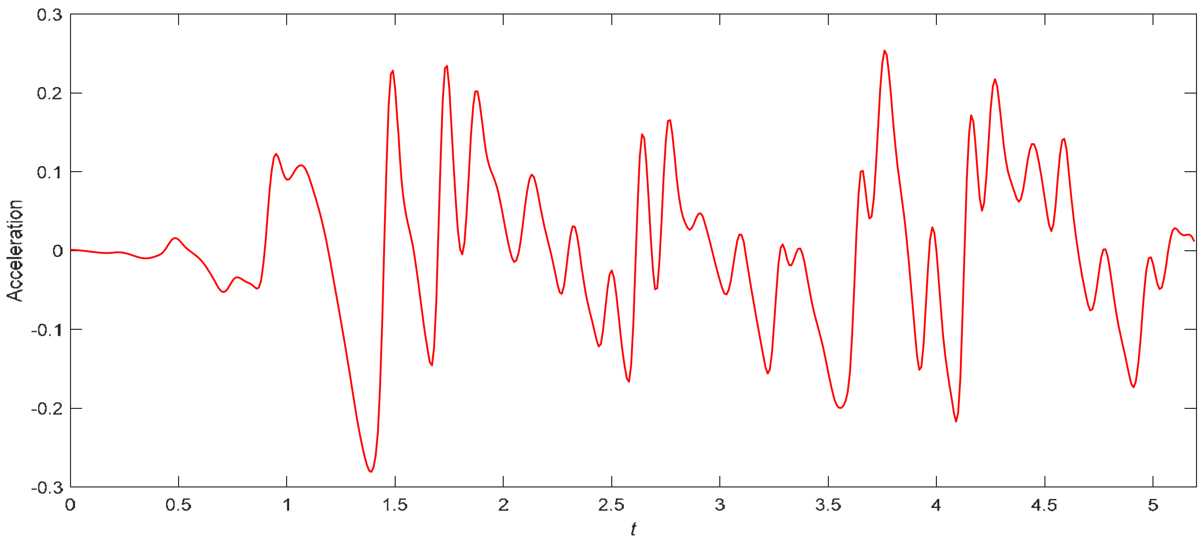


Fig. 1 Earthquake excitation

Eq. (17) can be also expressed with respect to the modal coordinates using the transformation $\underline{y}(t)=\Phi\underline{\xi}(t)$, as follows:

$$\ddot{\underline{\xi}}(t)+C^{*}\dot{\underline{\xi}}(t)+\Omega\underline{\xi}(t)=-\Phi^{T}M\underline{1}\ddot{y}_g(t) \tag{20}$$

where $C^{*}$ and $\Omega$ are two diagonal matrices with elements $2\zeta\omega_i$ and $\omega_i^2$, respectively. The state-space form is next constructed:

$$\dot{\underline{x}}(t)=A_c\underline{x}(t)+B_c p(t) \tag{21}$$

where $\underline{x}$ is state vector, $\dot{\underline{x}}$ is first derivative of the state, $\mathbf{A}_c$ is system state matrix and $\mathbf{B}_c$ is the input to state matrix given as:

$$x(t)=\begin{bmatrix}\xi(t)\\ \dot{\xi}(t)\end{bmatrix},\ A_c=\begin{bmatrix}\underline{0} & I\\ -\Omega & -C^*\end{bmatrix},\ B_c=\begin{bmatrix}\underline{0}\\ -\Phi M\underline{1}\end{bmatrix},\ p(\mathrm{t})=\ddot{y}_g(t) \tag{22}$$

The observation equation can also be written in the form:

$$\underline{d}(t)=G_c\underline{x}(t)+J_c p(t) \tag{23}$$

For absolute acceleration measurements $\underline{d}(t)$, the matrixes $G_c$ and $J_c$ are given by:

$$G_c=\left[-S_a\Phi\Omega\ -S_a\Phi C^*\right],\ J_c=\left[S_a(\underline{1}-\Phi\Phi^T M\underline{1})\right] \tag{24}$$

where $S_a$ is the selection matrix. Thus, the system of equations (23) and (24) can be applied to predict the acceleration measurements.

### 3.2 Results

The proposed method is now applied to the system mentioned above. Two cases are investigated in this section. The first one studies the problem of parameter estimation using the data features to formulate the likelihood, with truncated normal (TN) distribution assumed for the square of the discrepancy between the measured and model predicted response time histories (Case 1). The other one formulates the likelihood in a similar way but assumes an exponential (EXP) distribution for the square error, instead of a truncated Gaussian distribution. The exponential distribution is also used to explore the relationship between likelihood-informed algorithm and the likelihood-free ABC algorithm. All methods are compared with the conventional Bayesian formulation assuming normal (NORM) distribution for the prediction errors at each time instant to construct the likelihood.

Results are presented for simulated measurements that are generated for a nominal spring-mass chain model. To simulate the effect of model error, 5% Gaussian noise is added to the acceleration response time histories generated from the nominal model. The acceleration measurements from all ten DOF of the system are considered. For demonstration purposes, a single stiffness parameter is considered as the model parameter to be updated. This parameter included the stiffness of the first three springs in the spring-mass chain system.

Parameter estimation results along with their uncertainties (5 and 95% quantiles) are presented in Figs. 2 and 3 for different sampling rates $\Delta t$ ranging from $0.1\Delta t$ to $10\Delta t$ of the same time history. The number of the samples $N_D$ are decreased accordingly from $10N_D$ to $0.1N_D$. Specifically, results from the proposed truncated Gaussian distribution (TN) are compared with results obtained from the conventional Bayesian method. It should be noted that the different sampling rates chosen do not affect the information contained in the data. Both methods give almost the same MAP estimates for the structural model parameter (Fig. 2). However, uncertainty bounds are substantially different for the two methods. Specifically, from the results in Fig. 2, it becomes evident that the conventional Bayesian method gives very small uncertainties that decrease as the number of sampling points increase. The proposed method based on the data features provides much higher uncertainties that are independent on the number of data points used. This is consistent with intuition since the information contained in the acceleration time history is almost independent of the sampling rate used in this example.

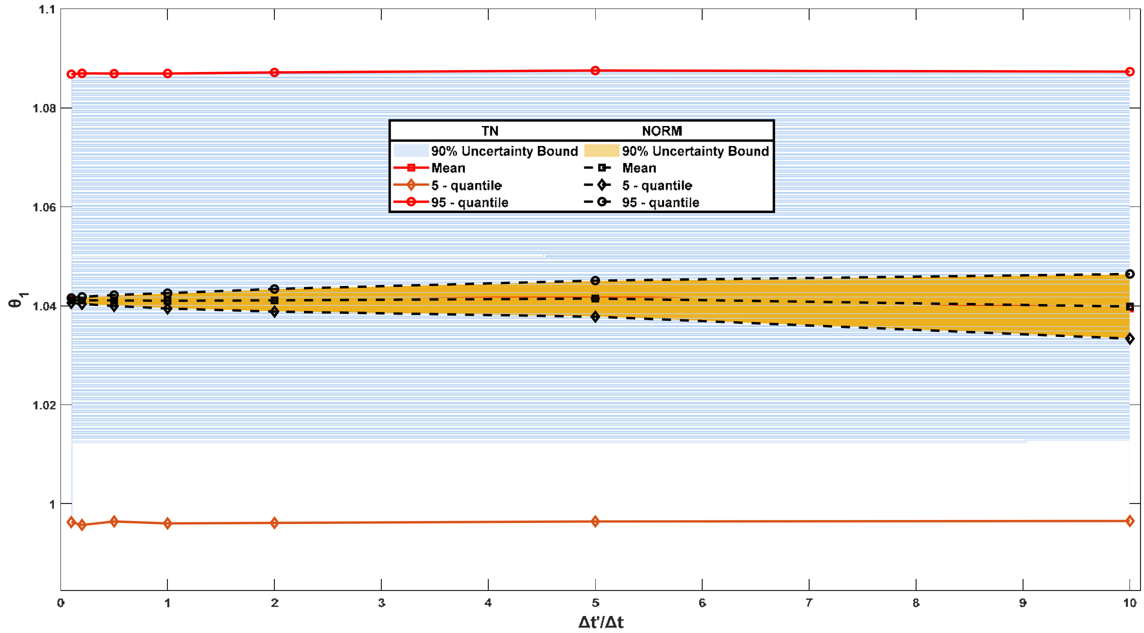


Fig. 2 Parameter estimation of $\theta_1$ using TN (Case 1) and NORM

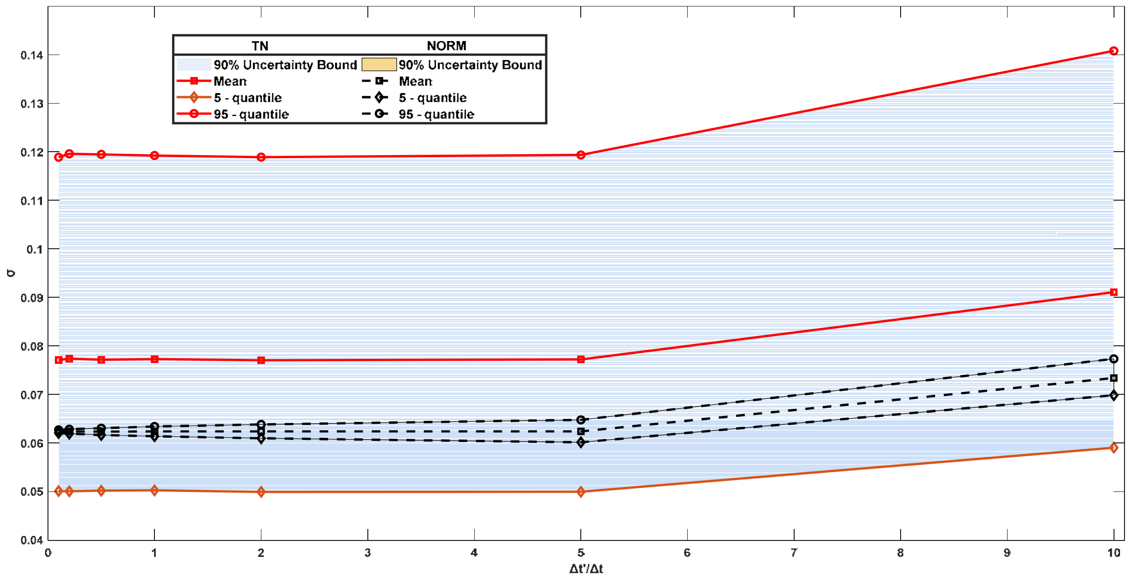


Fig. 3 Parameter estimation of $\sigma$ using TN (Case 1) and NORM

Next, parameter estimation results along with their uncertainties (5 and 95% quantiles) are compared in Fig. 4 for the TN case (Case 2), the conventional Bayesian method (NORM) and the ABC method. Again the sampling rates $\Delta t$ range from $0.1\Delta t$ to $10\Delta t$ of the same time history. The best estimate $\hat{\sigma}$ of the standard deviation is specified as the value $\sigma_{90}$ (90-quantile) obtained from EXP. Then the tolerance value $\delta$ in ABC algorithm can be calculated based on the formulation in Eq. (16). Although all three methods predict the same MAP estimate, the uncertainty bounds computed from the conventional Bayesian methods are again substantially smaller than the bounds computed from the other two methods. Also, the uncertainty in the model parameter decreases as the sampling rate increases which is contrary to intuition, since there is not extra information contained in the time history with higher sampling rate. The uncertainty predicted by the proposed likelihood-informed method is similar to the uncertainty estimated by the ABC method. Both methods (TN and ABC) provide uncertainty bounds that are almost independent on the sampling rate. The small discrepancies in the uncertainty bounds are due to the choice of the tolerance value in ABC. A slightly different tolerance can zero the discrepancy between the two methods.

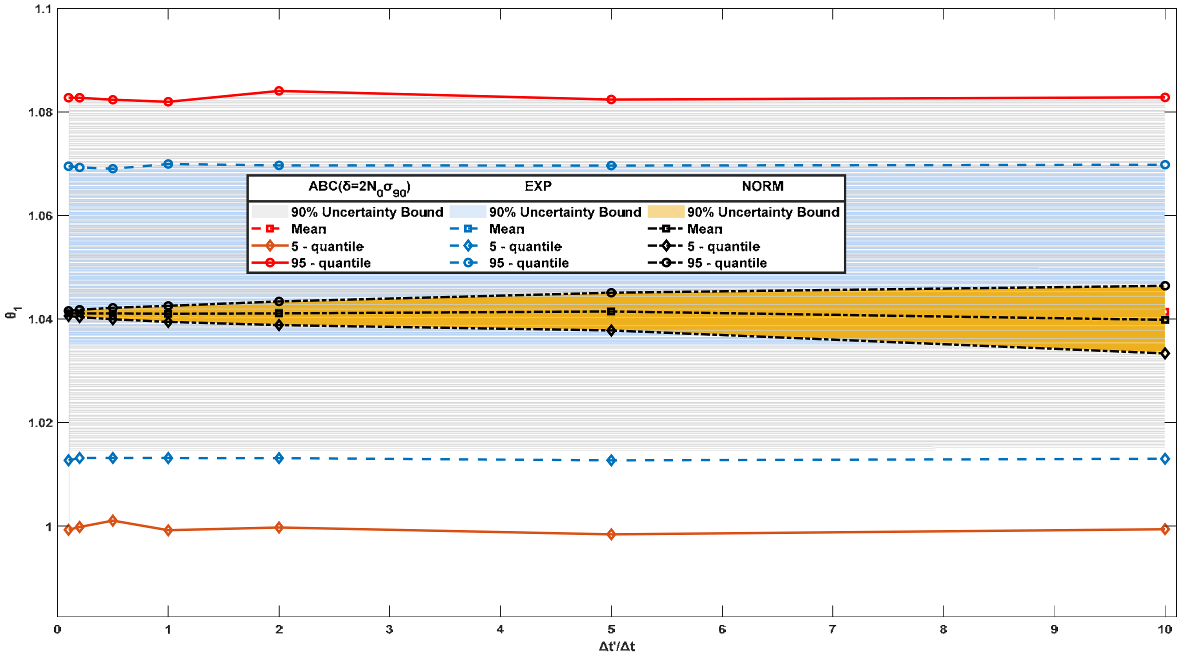


Fig. 4 Parameter estimation of $\theta_1$ using EXP (Case 2), NORM and ABC

## 4 CONCLUSIONS

A new formulation based on the data features for likelihood-informed Bayesian inference has been presented and discussed in this paper. The effectiveness of the proposed formulation has been demonstrated by a spring-mass chain model. The main conclusions of this work are:

- The proposed data-features likelihood-based Bayesian methodology correctly accounts for the uncertainty in the model parameters, making such uncertainty independent of sampling rate of the measured response time histories. In contrast, the uncertainty in the model parameters obtained from conventional Bayesian inference formulation depends on the sampling rate of the response time histories, despite the fact that the information contained in the response time history data is independent of the sampling rate.
- The proposed likelihood-informed Bayesian formulation provides results that are consistent with the ones obtained from likelihood-free ABC formulations.
- The proposed method applied herein to linear structural systems can also be extended to non-linear structural systems given response time history measurements.

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