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APPROXIMATE BAYESIAN MODAL ANALYSIS WITH PARTICLE-SWARM PROPOSALS

M.D. Champneys[1], **T.J. Rogers**[1]

[1]Dynamics research group, University of Sheffield
Mappin St, S Sheffield S1 4ET
{max.champneys,tim.rogers}@sheffield.ac.uk

Abstract. *Modal analysis has become the default framework for the consideration of structural dynamics. However, performing modal analysis within a Bayesian framework is made difficult by expensive or intractible likelihood functions. Approximate Bayesian computation sequential Monte Carlo (ABC-SMC) methods have proven to be effective at so-called likelihood-free inference tasks; situations whereby the likelihood distribution is either intractable or impractical to compute. The method approximates the true posterior distribution by tempering a threshold value within a rejection sampling scheme such that only parameters that provide increasingly good descriptions of the data are accepted. An important consideration in ABC-SMC is the choice of proposal distribution from which samples are drawn at each iteration. In the literature, several approaches are commonplace including Gaussian proposals fitted to samples from previous iterations and Gaussian perturbations on samples directly. In this work, an alternative proposal approach inspired by particle swarm optimisation is presented. The proposed approach is demonstrated and compared to Gaussian proposals in a case-study modal parameter identification problem from structural dynamics.*

Keywords: Modal analysis, ABC-SMC, Parameter identification, Uncertainty quantification, Structural dynamics

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1 Introduction

Parameter estimation tasks are pervasive in engineering analysis and beyond. In a linear structural dynamics setting, reduced-order models of complex structures are often obtained experimentally by modal analysis [1]. The salient parameters of the dynamics (the natural frequencies ω_n, damping ratios ζ_n and modeshapes Φ) are fitted numerically to measured data. The fitting to data can be performed in the time domain (displacements, accelerations, etc.) [2, 3], or in the frequency domain, by targeting the frequency response function (FRF) of the system [4, 5, 6]. Experimental modal analysis can essentially, be viewed as a parameter identification problem [1]. Over several decades, powerful methods for identification have been developed in other ways, including evolutionary computation [7]. However a thorough treatment of uncertainty quantification (both in terms of parameter uncertainty and model order) in modal analysis remains a key challenge. Although several promising directions have emerged, there remains a need to provide a practical quantification of the uncertainty in modal parameter identification.

A convenient framework for consideration of uncertainty in a parameter identification setting is to take a Bayesian viewpoint. For the task of modal identification and uncertainty quantification this will involve attempting to form the posterior distribution,

$$p(\omega_n, \zeta_n, \Phi | y) \tag{1}$$

over the modal parameters $\{\omega_n, \zeta_n, \Phi\}$ given some observed data y, accessible via Bayes rule. However, a key barrier to the development of uncertainty quantification schemes for modal analysis is that the data likelihood function,

$$p(y | \omega_n, \zeta_n, \Phi) \tag{2}$$

can be infeasible to compute. In the time domain, evaluation of the likelihood function would require many expensive simulations of the dynamics. In the frequency domain the likelihood functions are analytically very complex [8] which can hinder analysis. Without access to a likelihood function, one must instead rely on so-called *likelihood-free* learning models.

One powerful algorithm for performing parameter estimation in the likelihood free-setting is approximate Bayesian computation sequential Monte-Carlo with rejection sampling (hereafter ABC-SMC) [9, 10, 11]. In order to keep this paper compact a brief introduction to ABC-SMC in the context of modal parameter identification is given below, however the interested reader is directed to [10] for a more thorough treatment of SMC methods.

The aim of ABC-SMC is to approximate the posterior distribution sequentially by tempering a threshold value ϵ over a distance $d(\hat{y})$ function between the observed data y and a the model predictions $\hat{y} = \mathcal{M}(\theta)$ approximated by sampled particles θ_i, $i \in 1, N$.

The ABC-SMC algorithm begins with a set of N particles, sampled from a prior distribution over the parameters.

$$\theta^0 \sim p(\omega_n, \zeta_n, \Phi) \tag{3}$$

The algorithm proceeds for a number of rejection sampling waves, sampling a further N particles,

$$\theta^{j+1} \sim \pi(\theta | \theta^{(j)}, d(\theta^{(j)})) \tag{4}$$

whereby $\pi(\cdot)$ is defined as the *proposal* distribution. Crucially, only particles that satisfy the inequality,

$$d(\mathcal{M}(\theta^{(j)})) < \epsilon_j \tag{5}$$

are accepted into posterior approximation in the next wave. The ABC-SMC algorithm continues in this manner, tempering the value of ϵ towards some small value until a stopping criteria is met. As $\epsilon \to 0$ and $N \to \infty$, the posterior approximation will contain the true posterior.

2 Choice of proposal distribution

A key factor in the efficiency of the ABC-SMC scheme is the choice of the proposal distributions at each wave. Many proposal methods are applicable (it is required that the proposal scheme be ergodic). However, inefficient proposal distributions can lead to slow convergence and low acceptance rates in the rejection sampler, dramatically increasing the computational cost of the method.

A common approach for the specification of the proposal distribution in ABC methods is to naively draw from a Gaussian distribution over the particles from the previous wave. This can be done globally (by fitting a Gaussian to the particles in the previous wave), or as a random walk on the particles as in population Monte Carlo (ABC-PMC) methods [12]. However these methods can be inefficient at exploring probable regions of the parameter space, an issue that only becomes more prevalent as the problem dimension increases.

One particular challenge that pertains to modal parameter identification is that the order of the modes in the distance function does not affect its value. This effect can be attributed the fact that the FRF H_{ij} is a simple linear superposition of the contributions of the n individual modes,

$$H_{ij}(\omega) = \sum_{r}^{n} \frac{\phi_i^{(r)} \phi_j^{(r)}}{-j\omega^2 + 2j\zeta\omega_n\omega_n + \omega_n^2} \tag{6}$$

This order-invariance manifests as multimodality (since the ω_n, ζ_n, etc. can appear in any order in θ) in the posterior distribution.

2.1 Proposed approach

The main contribution of this study is a new proposal scheme for ABC-SMC based on particle swarm dynamics (ABC-PSO). The proposed approach is inspired by the efficacy of heuristic search algorithms in locating the global optima of multimodal optimisation problems using only evaluations of an objective function. Of such heuristics, the particle-swarm algorithm [13, 14] has proven to be extremely performant in a wide range of fields, and countless variants have been proposed [15, 16, 17]. Using heuristic methods to inform proposal schemes in ABC is not in itself a new idea. In [18] a differential evolution based approach (ABCDE) is taken, and in [19] a Markov-Chain Monte Carlo based image tracker is proposed that leverages particle-swarm optimisation. However, the proposed approach in this study can be considered a drop-in replacement for other particle proposal schemes in ABC-SMC, whereas others are Monte-Carlo based scheme.

Particle-swarm methods perform optimisation by tracking the dynamics of a number of particles moving around the parameter space governed by a stochastic differential equation (SDE). The motions of the particles are governed by attraction forces that draw particles towards a number of promising regions for exploration. Common choices for the attraction points in the

Table 1: Hyperparameters of the PAO algorithm used in this study.

Parameter	Description	Value
m_{PSO}	Particle inertia	1
ζ_{PSO}	Particle damping ratio	0.2
k_{local}	Local particle attractor stiffnesses	1
k_{global}	Global particle attractor stiffnesses	1
Δt_{PSO}	PAO Integration time interval	1
q	Additive stochasticity scaling factor	1

particle-swarm algorithm are the *local best*, defined as the lowest objective function score in the history of a single particle and *global best*, defined as the lowest objective function score across all particles. To prevent collapse to these values, the motions of the particles in the swarm are regulated by inertial, viscous damping and stochastic terms.

3 Methodology

The ABC-PSO scheme presented in this study generates samples $\theta^{(j+1)}$ at wave $j+1$ by sampling from a single iteration of a modified particle swarm based on the values of samples $\theta^{(j)}$. Overall, the motions of each element of each particle in each wave are governed by the differential equation,

$$m_{\text{PSO}}\ddot{\theta}' + 2\sqrt{m_{\text{PSO}}k'}\zeta_{\text{PSO}}\dot{\theta}' + k'\theta' = qe \tag{7}$$

whereby,

$$k' = k_{\text{local}} + k_{\text{global}} \tag{8}$$

is the overall effective stiffness,

$$\theta' = \theta - \frac{k_{\text{local}}\theta_{\text{local}} - k_{\text{global}}\theta_{\text{global}}}{k'} \tag{9}$$

is the relative position of each particle to the overall effective attractor between the local θ_{local} and global θ_{global} best, and,

$$e \sim \mathcal{N}(0, (\theta_{\text{local}} - \theta_{\text{global}})^2) \tag{10}$$

To solve for the particle positions, one is able to adopt a number of numerical methods to integrate the dynamics forward in time. Since the stochasticity is additive and Gaussian in the above equation, an exact discretisation is taken here [20].

The values of all parameters used in the particle swarm implementation used in this study are collected in Table 1.

In order to demonstrate that the performance of the ABC-PSO approach, the ABC-PSO method is compared to a standard ABC-SMC approach with a Gaussian proposal fitted to the particles in the previous wave. In order to overcome the mode-order invariance, the particles in the Gaussian proposal ABC-SMC are sorted prior to the evaluation of the covariances. However, this is not required in the proposed particle-swarm approach.

For target data, a simulated FRF from a three degree-of-freedom (DOF) system is constructed. Rather than use the Full FRF matrix for identification, only the first row is taken. This is more indicative of the type of single-input tests that are used for modal identification

Table 2: Parameters of the simulated 3dof modal dynamics.

n	ω_n (Hz)	ζ_n (%)	ϕ_n
1	2.42	1.21	[-0.5,-0.7,-0.5]
2	4.47	2.23	[-0.7,0,0.7]
3	5.84	2.92	[-0.5,0.7,-0.5]

Table 3: Parameters relating to the ABC rejection sampling scheme used in this study.

Parameter	Description	Value
N	Number of particles	1000
n_ω	Number of spectral lines	1000
ϵ_0	Initial acceptance threshold	500
$\Delta\epsilon$	Epsilon decay percentile	10

in practice. For the readers convenience, the modal properties of the simulated system are collected in Table 2.

The distance measure d was chosen to be an $\mathcal{L}_2$ norm on the real and the imaginary parts of the FRF independently,

$$d(\theta) = ||\Re(\mathcal{M}(\theta) - f)||^2 + ||\Im(\mathcal{M}(\theta) - f)||^2 \tag{11}$$

where f is the measured FRF data. Table 3 contains all parameters relating to the ABC-SMC implementation.

At the end of each iteration, a new value for the threshold parameter ϵ is selected. In the literature, a number of methods for selecting the next threshold have been proposed [21], but in this study, the new value is simply chosen as the 20^{th} percentile of the distance metric values from the accepted particles in the preceding wave.

The SMC schemes described in this paper are iterated until a convergence criteria is met. Here, iterations are halted after the value of the threshold parameter falls below 2×10^{-2}.

4 Results

The proposed particle-swarm approach is compared to ABC-SMC with a Gaussian proposal scheme on the benchmark modal parameter identification task with the configurations above as described.

For the prior distribution a Gaussian distribution is assumed independently over each parameter. Histograms of samples from the prior distribution are plotted in Figure 1. To remove the mode-order independence from the data, the samples are sorted modally by natural frequency in the plotted data.

The model draws from the prior for the first DOF ($H_{11}(\omega)$) are plotted in Figure 2. It is promising that the prior draws contain the truth as well as considerable variance.

Once the convergence criteria was met, the posterior distributions are collected from the

Table 4: Parameters of the prior distribution over the modal parameters.

Parameter	μ	σ
ω_n	4.0	3.0
$\log \zeta_n$	0.55	1.24
ϕ_{nk}	0.0	1.0

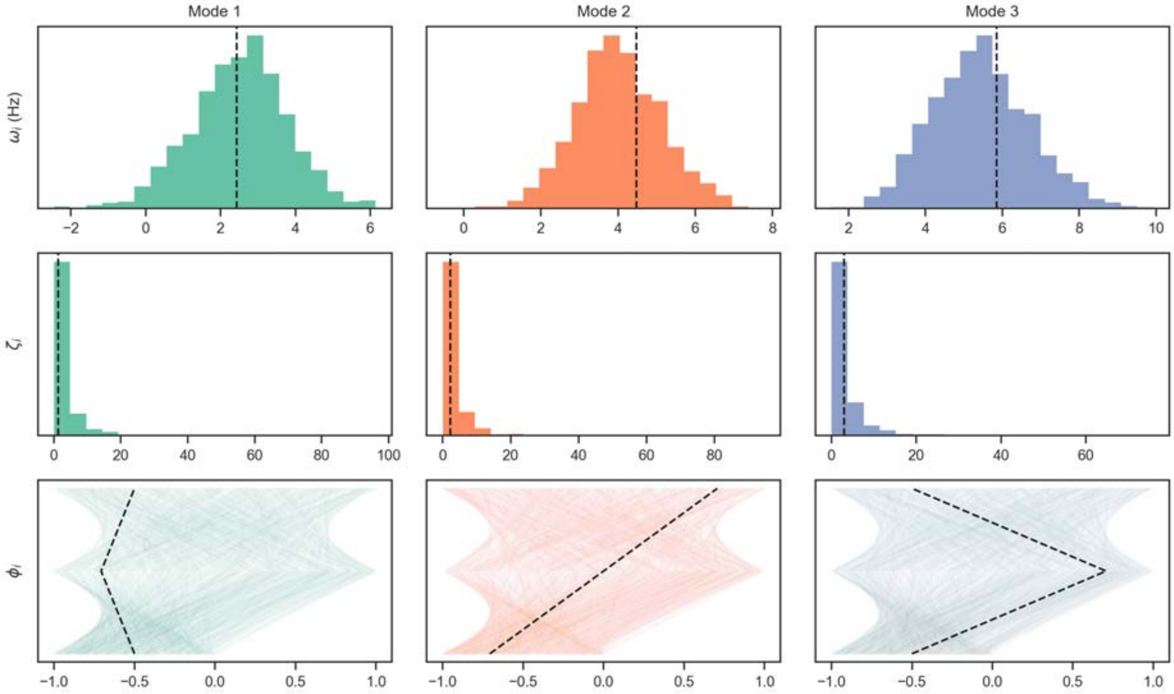

Figure 1: Parameter distributions under the prior used for both ABC schemes considered in this paper.

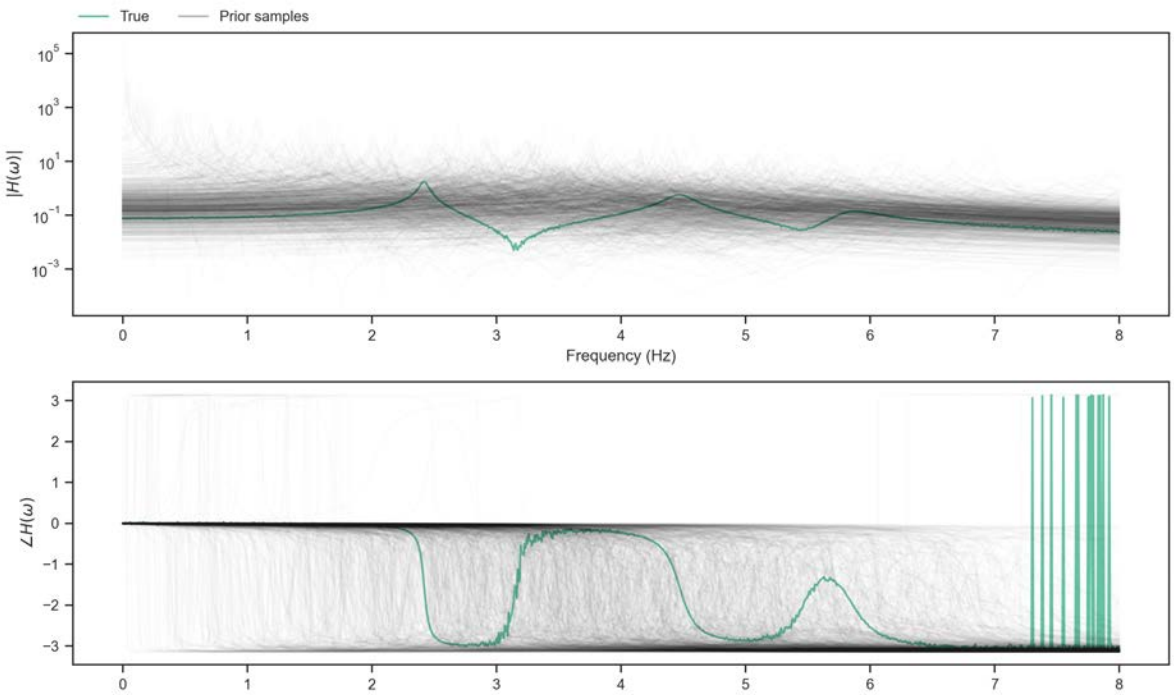

Figure 2: Samples drawn from the prior distribution over the modal parameters compared to the simulated target data.

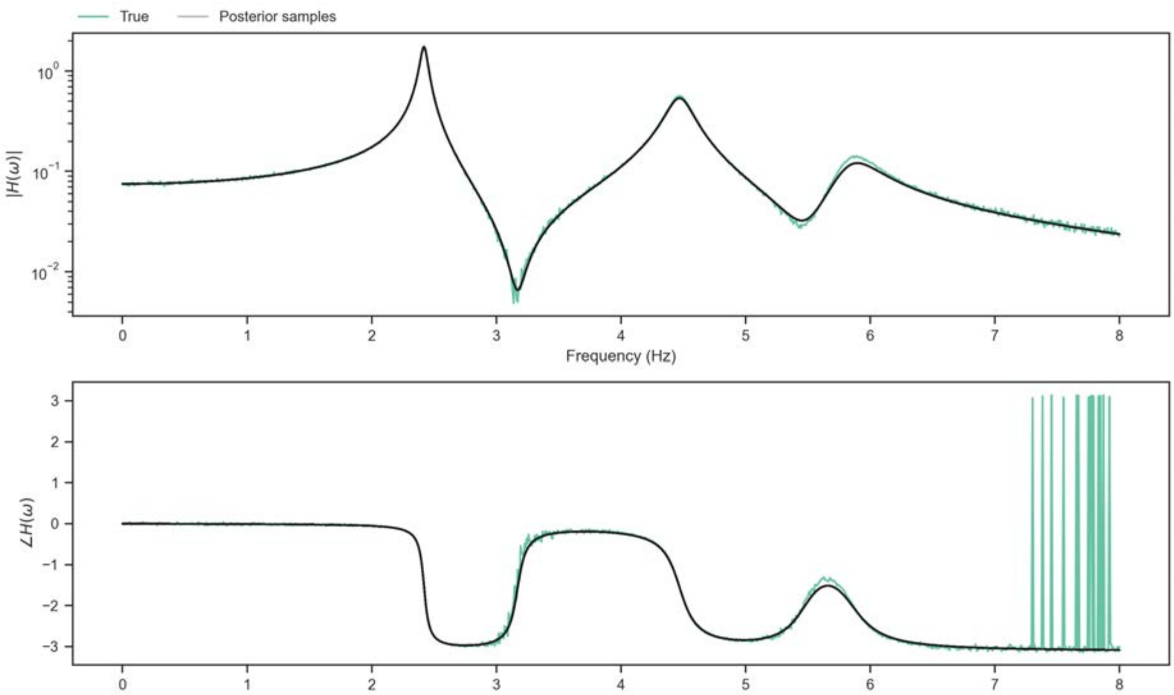

Figure 3: Samples drawn from the ABC-SMC posterior distribution over the modal parameters compared to the simulated target data.

accepted particles in the final wave of the ABC scheme. Posterior FRFs ($H_{11}(\omega)$) from the ABC-SMC approach are plotted in Figure 3. As can be seen in the figure, there is excellent agreement with the simulated data ($\bar{d} = 0.084$) although the damping on the third mode appears to have been slightly over-estimated.

Parameter-wise histograms are plotted for the posterior distribution from the ABC-SMC scheme in Figure 4. The parameters have been identified to a very high degree of confidence extremely close to the true values. In most cases the true parameters have been identification to well-within the sampling precision of the data with the possible exception of the damping ratio for the third mode that has been over estimated by an absolute value of around 0.5.

The posterior samples from ABC-PSO are plotted in Figure 5. Once again, the agreement to the simulated data is excellent ($\bar{d} = 0.020$).

Although the performance of the two approaches is similar in terms of the convergence to the true modal parameters, it is instructive to also compare the sampling efficiency. In figure 7, the acceptance rates in the rejection sampling step is compared. A higher acceptance rate corresponds to a reduction in the total number of particles that must be sampled during the run of the SMC scheme. It is clear from the figure that the proposed ABC-PSO scheme outperforms the ABC-SMC approach in this regard.

It is also worth comparing the rate at which the acceptance threshold ϵ decays during the run. Figure 8 compares the decay of the this parameter on a logarithmic scale. In the figure, it can be seen that the ABC-PSO scheme reaches a lower acceptance threshold, faster than the ABC-SMC approach. Although this is the result of only a single run of the optimisation, it might be reasoned that this is because the particle swarm is able propose samples from promising parts of the parameter space more readily that is possible with naively fitted Gaussian proposals.

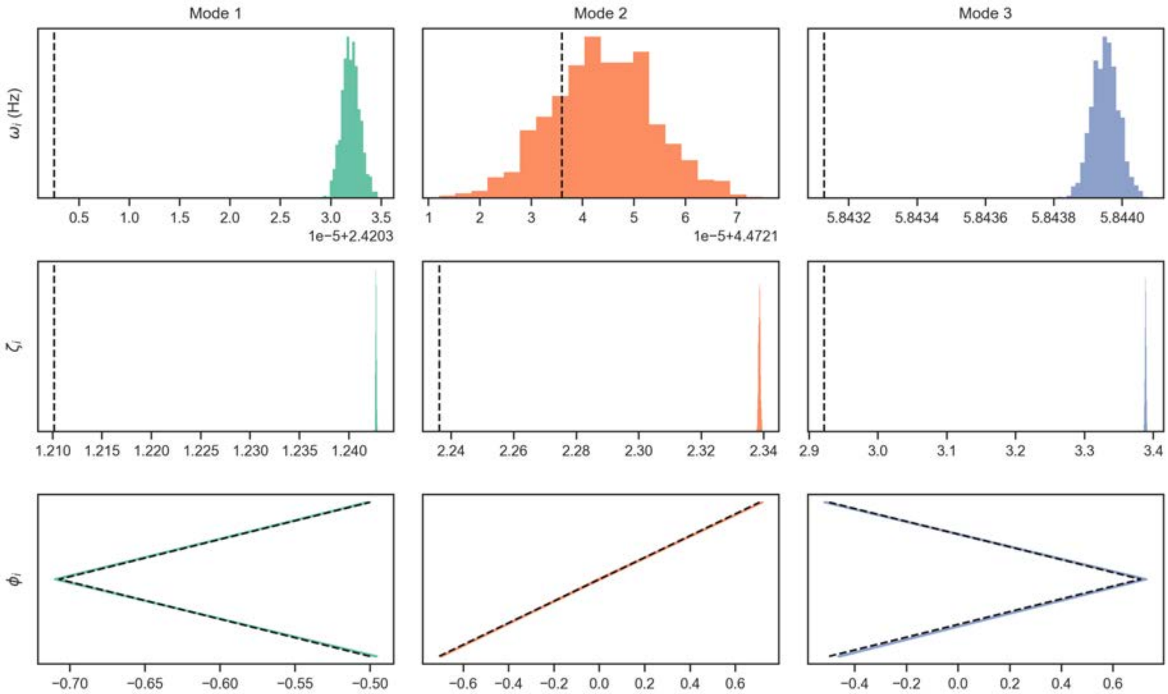

Figure 4: Parameter distributions from the posterior distribution of the ABC-SMC scheme, dashed lines are the true values.

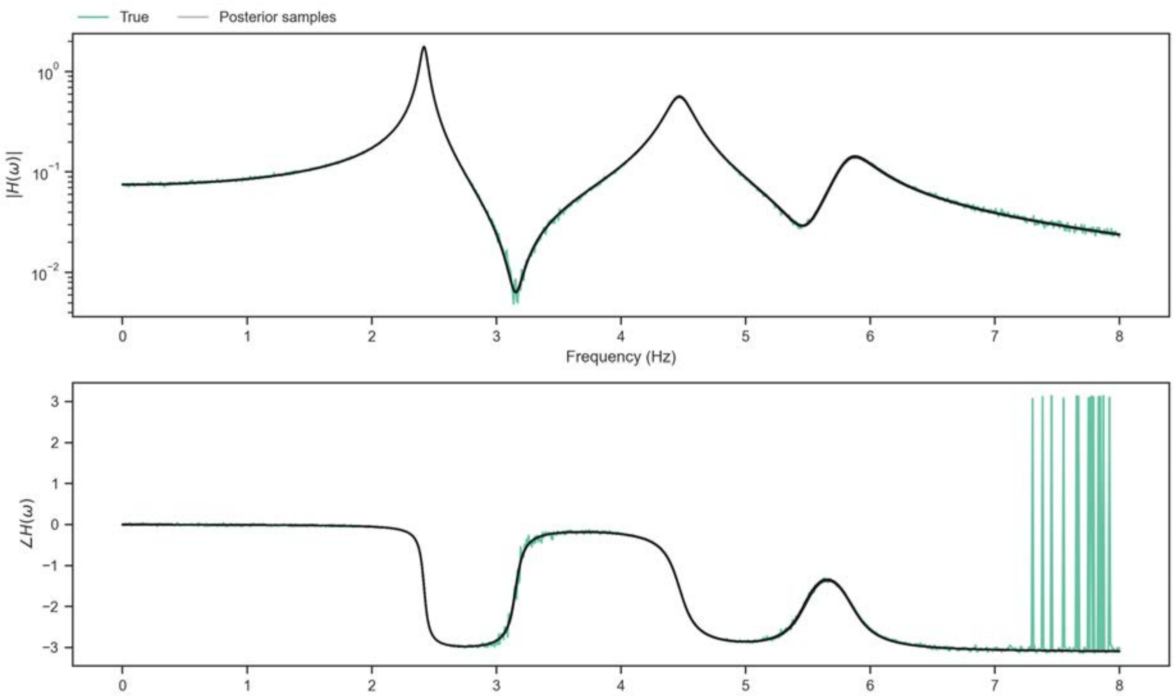

Figure 5: Samples drawn from the ABC-PSO posterior distribution over the modal parameters compared to the simulated target data.

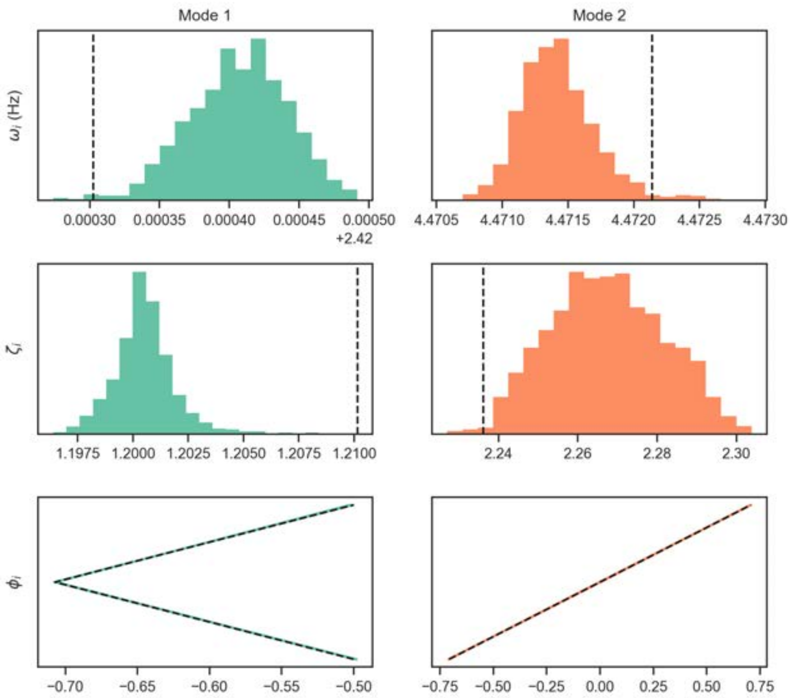

Figure 6: Parameter distributions from the posterior distribution of the ABC-PSO scheme, dashed lines are the true values.

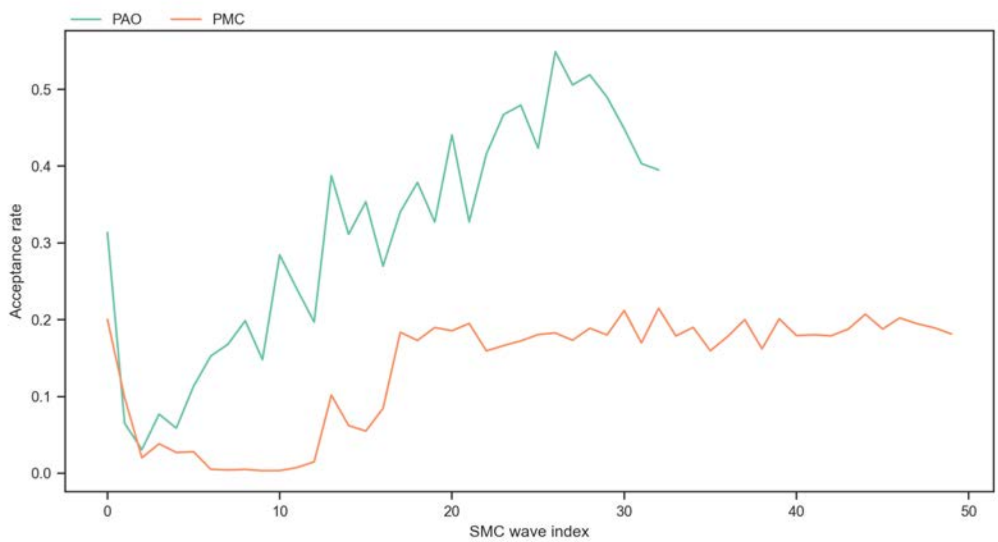

Figure 7: Acceptance rates of the ABC schemes plotted per wave.

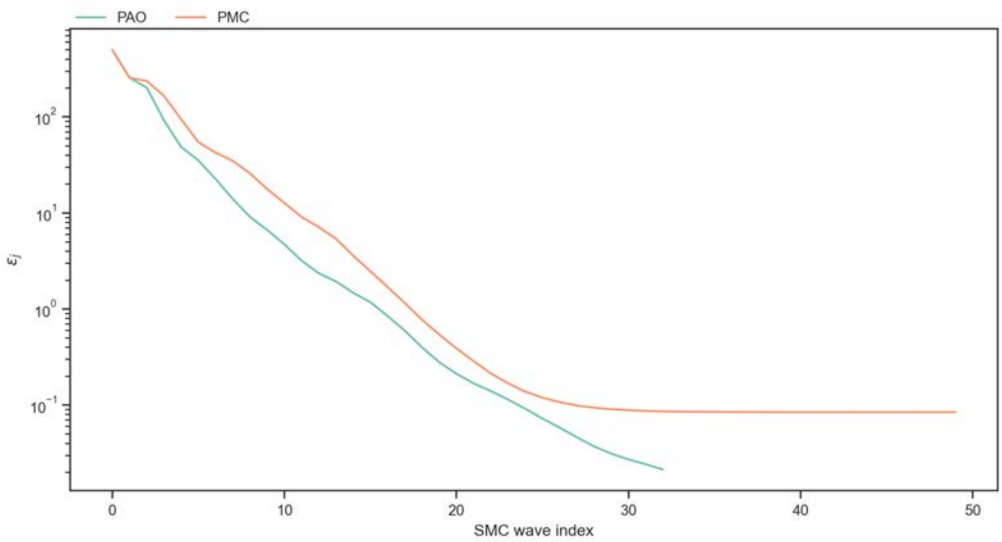

Figure 8: Threshold levels plotted per wave.

5 Conclusions

Overall, the particle-swarm approach presented in this paper shows considerable promise as a proposal scheme in ABC-SMC for the identification of modal parameters. In a benchmark comparison study, the proposed approach outperformed a standard ABC-SMC with Gaussian proposal both in terms of the average distance to the data in the posterior and (perhaps more importantly) the number of samples required to reach the threshold tolerance.

More generally, this work demonstrates that the intersection of heuristic optimisation methods and SMC offers an interesting research vein for further investigation.

Uncertainty quantification remains a key challenge in quantifying uncertainty in modal analysis. Interesting avenues for further work include extending the results of this paper to experimental data, consideration of importance sampling in SMC-PSO and the consideration of automatic model order detection.

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