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ON THE CUMULATIVE DISTRIBUTION FUNCTION OF OUTPUT RANDOM VARIABLES IN THE STUDY OF DYNAMIC SYSTEMS

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Abstract. *The method presented in the article enables us to easily determine the cumulative distribution function of certain relevant quantities that depend on the solution of a dynamic process, which includes some random variables as input parameters. The article provides three examples. In the first two examples, the obtained solution is compared with the explicit one. The third example deals with a masonry tower that was subjected to an acceleration recorded during the 2013 Fivizzano earthquake. Assuming that Young's modulus of the material and the PGA of the earthquake are random variables whose probability density function is known, the cumulative distribution functions of both the attained maximum tower top displacement and the maximum cracked volume are determined.*

Keywords: Dynamic system, Stochastic process, Uncertainty Quantification.

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1 INTRODUCTION

When studying evolution problems, such as structures subjected to dynamic loads, it's often necessary to consider uncertainties that affect certain parameters characterizing the geometry, mechanical properties of materials, and external actions. These parameters are thus represented by several random variables Θ_i, defined on an appropriate probability space Λ, and their (joint) probability density function $p(\theta)$ is assumed to be known. As a result, the solution of the dynamic problem, which depends on time and these random variables, becomes a stochastic process. In such cases, it's necessary to determine the time evolution of the probability distribution of some output quantity Z, that, being defined in terms of the solution to the dynamic problem, is itself a stochastic process.

Besides using the Monte Carlo Method, which is generally popular but requires significant computational effort, the problem can also be addressed using the generalized density evolution method [1]. This method involves solving a partial differential equation, and its formulation requires knowledge of the time derivative of Z, which is calculated while solving the dynamic system. The method was implemented in the MADY code [2] and applied to the study of some masonry structures [3], [4].

More recently, the authors have approached the problem from a slightly different perspective [5]. They proved that the evolution of the cumulative distribution function of Z (which is always more regular than the corresponding density function) can also be determined by solving an equation similar to the one proposed in [1].

On the other hand, when it is sufficient to know this distribution function at a limited number of instances, it can be simply determined by calculating several integrals on a suitable subset of Λ. This method has the further advantage of not requiring knowledge of the time derivative of Z, and its potential is explored in this paper.

The article then presents three examples. In the first two, the method is applied in situations where it is also possible to determine the explicit solution, which is then compared with the numerical one. In the third example, it's assumed that Young's modulus and the PGA of the earthquake are random variables whose probability density function is known. Then, the cumulative distribution function of both the maximum displacement at the top of the tower and the maximum cracked volume attained during the earthquake is determined.

2 BACKGROUND AND NOTATIONS

Let $(\Omega, \mathcal{A}, P)$ be a probability space. Here Ω is the set of the *outcomes*, $\mathcal{A}$ is the σ-algebra on Ω made of all the *events* and $P : \mathcal{A} \to [0, 1]$ is a probability measure, i.e. a positive measure on $(\Omega, \mathcal{A})$ such that $P(\Omega) = 1$. Moreover, let $\mathtt{R}^n$ and $\mathcal{B}(\mathtt{R}^n)$ be the n-dimensional Euclidean space and its corresponding Borel σ-algebra, respectively.

A map $\boldsymbol{X} : \Omega \to \mathtt{R}^n$ is said to be a (*vector*) *random variable* if it is $\mathcal{B}(\mathtt{R}^n)$-measurable, i.e. if

$$\{\boldsymbol{X} \in B\}, \text{ for each } B \in \mathcal{B}(\mathtt{R}^n),$$

where, as usual, $\{\boldsymbol{X} \in B\}$ denotes $\boldsymbol{X}^{-1}(B)$. Let μ_X denote the *law* of $\boldsymbol{X}$ (or the *image* of measure P under $\boldsymbol{X}$) [6], i.e. the measure defined on $(\mathtt{R}^n, \mathcal{B}(\mathtt{R}^n))$ by

$$\mu_X(B) = P(\{\boldsymbol{X} \in B\}), \text{ for each } B \in \mathcal{B}(\mathtt{R}^n),$$

then the function $F_X : \mathtt{R}^n \to [0, 1]$ defined by

$$F_X(a_1, a_2, ..., a_n) = \mu_X(\{\boldsymbol{x} \in \mathtt{R}^n : -\infty < x_i \leq a_i, i = 1, 2, ..., n\}),$$

is called (*cumulative*) *distribution function* (CDF) of $\boldsymbol{X}$.

Let $\mathcal{L}^n$ be the Lebesgue measure on $(\mathtt{R}^n, \mathcal{B}(\mathtt{R}^n))$. Measure μ_X is said to be *absolutely continuous* (with respect to $\mathcal{L}^n$) if $\mu_X(B) = 0$ for every set $B \in \mathcal{B}(\mathtt{R}^n))$ with $\mathcal{L}^n(B) = 0$. In this case, the Radon-Nikodym theorem [6] guarantees the existence of a positive integrable function p_X such that

$$\mu_X(B) = \int_B p_X(\boldsymbol{x})dx \tag{1}$$

for every $B \in \mathcal{B}(\mathtt{R}^n))$. Function p_X is called the (*joint) probability density function* of the vector random variable $\boldsymbol{X}$ and it holds

$$F_X(x_1, x_2, ...x_n) = \int_{-\infty}^{x_1} .... \int_{-\infty}^{x_n} p_X(\xi_1, \xi_2, ..., \xi_n) d\xi_1 d\xi_2 ... d\xi_n.$$

Let $\boldsymbol{X} : \Omega \to \mathtt{R}^n$ be a vector random variable and $g : \mathtt{R}^n \to \mathtt{R}^m$ a Borel function; then $\boldsymbol{Y} = g \circ \boldsymbol{X}$ is again a vector random variable and, for each $B \in \mathcal{B}(\mathtt{R}^m)$,

$$\mu_Y(B) = \mu_X(g^{-1}(B)). \tag{2}$$

If $\boldsymbol{X}$ has a probability density function p_X, then

$$\mu_Y(B) = \mu_X(g^{-1}(B)) = \int_{g^{-1}(B)} p_X(\boldsymbol{x})dx \tag{3}$$

by (1) and (2).

A *stochastic process* $\{\boldsymbol{X}_t, t \in D\}$, with $D = [0, \bar{t}]$ a real interval, is a family of vector random variable indexed by a parameter t, defined on a common probability space $(\Omega, \mathcal{A}, P)$ and all with their values in $(\mathtt{R}^n, \mathcal{B}(\mathtt{R}^n))$, which is called the *state space*. By definition, for each $t \in D$, $\boldsymbol{X}_t$ is an $\mathcal{A}$-misurable function and, for each $\omega \in \Omega$, $\{\boldsymbol{X}_t(\omega), t \in D\}$ is a function defined in D that is called *sample function, realizzation* or *trajectory* of the process.

Let $U \subset \mathtt{R}^n$ be an open set and $G : U \times D \to \mathtt{R}^n$ a continuous map. Let us consider the system of ordinary differential equations

$$\dot{\boldsymbol{x}} = G(\boldsymbol{x}, t), \qquad \boldsymbol{x}(0) = \boldsymbol{x}_0, \qquad t \in D. \tag{4}$$

In this form, numerous evolution problems can be expressed, including in particular the equation of motion of a structure that has been discretized with finite elements. If randomness is present, coming from the initial conditions, the excitations, or the properties of the system, a random equation can be written as

$$\dot{\boldsymbol{x}} = G(\boldsymbol{\theta}, \boldsymbol{x}, t), \tag{5}$$

where $\boldsymbol{\theta} = (\theta_1, \theta_2, ...\theta_m) \in \Lambda \subset \mathtt{R}^m$ is the vector of all random parameters, i.e. the values assumed by the vector random variable $\boldsymbol{\Theta} : \Omega \to \Lambda$, that is supposed to have a (joint) probability density function $p_{\boldsymbol{\Theta}}$.

If the (deterministic) problem (4) is well posed, then for each choice of $\boldsymbol{\theta}$ (and $\boldsymbol{x}_0$), equation (5) has one and only one solution

$$\boldsymbol{x} = \mathbf{H}(\boldsymbol{\theta}, t),$$

with $\mathbf{H} : \Lambda \times D \to \mathtt{R}^n$ a suitable smooth function. (For sake of simplicity, we omit to explicitly indicate the dependence of $\mathbf{H}$ on $\boldsymbol{x}_0$).

In applications, we are interested in considering stochastic processes of the type

$$Z_t(\boldsymbol{\theta}) = Z(\boldsymbol{\theta}, t) = \psi \circ \mathbf{H}(\boldsymbol{\theta}, t), \qquad Z(\boldsymbol{\theta}, 0) = \psi(\boldsymbol{x}_0) = z_0, \tag{6}$$

with $\psi : \mathbb{R}^n \to \mathbb{R}$ a deterministic smooth function, and in determining the distribution function $F_Z(z,t)$ of Z_t, for different values of t. Of course, the method can also be applied in the case where Z is independent of t.

As proved in [5], it turns out

$$F_Z(z,t) = \int_{\{Z_t \le z\}} p_{\boldsymbol{\Theta}}(\boldsymbol{\theta}) d\boldsymbol{\theta}, \tag{7}$$

where

$$\{Z_t \le z\} = \{\boldsymbol{\theta} \in \Lambda : z - Z_t(\boldsymbol{\theta}) \ge 0\}. \tag{8}$$

For each t, relation (7) can also be obtained directly from (3), with $\boldsymbol{X} = \boldsymbol{\Theta}, \boldsymbol{Y} = Z_t \circ \boldsymbol{\Theta}$ and $B = (-\infty, z]$. Moreover, this result can be generalized to the case where $\boldsymbol{Z}_t$ is a vector-valued function, that is

$$F_{\boldsymbol{Z}}(\boldsymbol{z},t) = \int_{\{\boldsymbol{Z}_t \le \boldsymbol{z}\}} p_{\boldsymbol{\Theta}}(\boldsymbol{\theta}) d\boldsymbol{\theta}, \tag{9}$$

where

$$\{\boldsymbol{Z}_t \le \boldsymbol{z}\} = \{\boldsymbol{\theta} \in \Lambda : \boldsymbol{z} - \boldsymbol{Z}_t(\boldsymbol{\theta}) \ge 0\} \tag{10}$$

and the inequalities must be interpreted componentwise.

In [5] the relation (7) has been used to prove that the evolution over time of the CDF of Z_t can be determined by solving a first-order partial differential equation, for each value of $\boldsymbol{\theta}$. Precisely, let be $N = \mathbb{R} \times \Lambda \times D, G(z, \boldsymbol{\theta}, t) = z - Z(\boldsymbol{\theta}, t)$ and χ_K the indicator function of the region

$$K = \{(z, \boldsymbol{\theta}, t) \in N : G \le 0\}.$$

Then, χ_K satisfies the linear partial differential equation

$$\frac{\partial \chi_K(z, \boldsymbol{\theta}, t)}{\partial t} + \frac{\partial Z(\boldsymbol{\theta}, t)}{\partial t} \frac{\partial \chi_K(z, \boldsymbol{\theta}, t)}{\partial z} = 0 \tag{11}$$

with the initial condition

$$\chi_K(z, \boldsymbol{\theta}, 0) = \begin{cases} 1 & \text{if} \quad z \ge z_0, \\ 0 & \text{otherwise.} \end{cases} \tag{12}$$

Once χ_K has been calculated, F_Z is obtained from the relation

$$F_Z(z,t) = \int_{\Lambda} \chi_K(z, \boldsymbol{\theta}, t) p_{\boldsymbol{\Theta}}(\boldsymbol{\theta}) d\boldsymbol{\theta}. \tag{13}$$

In the following examples, the equation (7) will be used to directly determine the CDF of Z_t, in a limited number of instants.

3 EXAMPLES

3.1 SDOF System

Let us consider the equation of motion of a mass-spring system [1]

$$\ddot{x} + \theta^2 x = 0, \qquad x(0) = x_0, \qquad \dot{x} = 0 \tag{14}$$

where θ is the value of a random variable Θ that is uniformly distributed on the interval $\Lambda = [5\pi/4, 7\pi/4]$. The solution to (14) is the stochastic process

$$X_t(\theta) = x_0 cos(\theta t). \tag{15}$$

Setting $Z_t(\theta) = X_t(\theta)/x_0$, the corresponding distribution function is

$$F_Z(z,t) = \frac{2}{\pi}\int_{\{cos(\theta t)\leq z\}} d\theta. \tag{16}$$

If we limit ourselves to considering the case in which $t \in [4/5, 8/5]$, we obtain

$$F_Z(z,t) = \begin{cases} 0 & \text{if } z \leq cos(5\pi t/4), \\ \frac{2(2\pi - cos^{-1}(z))}{\pi t} - \frac{5}{2} & \text{if } cos(5\pi t/4) < z \leq cos(7\pi t/4), \\ 1 & \text{if } z > cos(7\pi t/4). \end{cases} \tag{17}$$

The calculation was done for $t = .9s$, $t = 1s$, and $t = 1.1s$, and the obtained results are shown in Fig. 1.

In the following example, equation (7) is applied to a case where Z is independent of time.

3.2 Traffic waves

Although the traffic model used here is very simple, it captures many of the qualitative and quantitative features of real traffic flow.

Let be $\rho(x,t)$ the car density, i.e. the number of cars per unit length of the road (in a unique lane), $v = v(\rho(x,t))$ the car velocity at point x and time t, and $q = \rho v(\rho)$ the car flowrate. The equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial q}{\partial x} = 0 \tag{18}$$

expresses the "conservation of cars" [7], [8]. Indicating with $c(\rho) = \frac{d}{d\rho}(\rho v)$ the kinematic wave speed, we obtain the first-order quasi-linear partial differential equation

$$\frac{\partial \rho}{\partial t} + c(\rho)\frac{\partial q}{\partial x} = 0. \tag{19}$$

Denoted by ρ_m the maximum allowed value of ρ (i.e. the value for which the cars are bumper to bumper), we make the "constitutive hypothesis"

$$v(\rho) = -\frac{v_0}{\rho_m}(\rho - \rho_m) \tag{20}$$

where v_0 is the velocity corresponding to $\rho = 0$ (i.e. the maximum permitted speed on the road). Finally, we assume the initial condition

$$\rho_0(x) = \rho(x,0) = \frac{\rho_L + \rho_R e^{x/L}}{1 + e^{x/L}}, \tag{21}$$

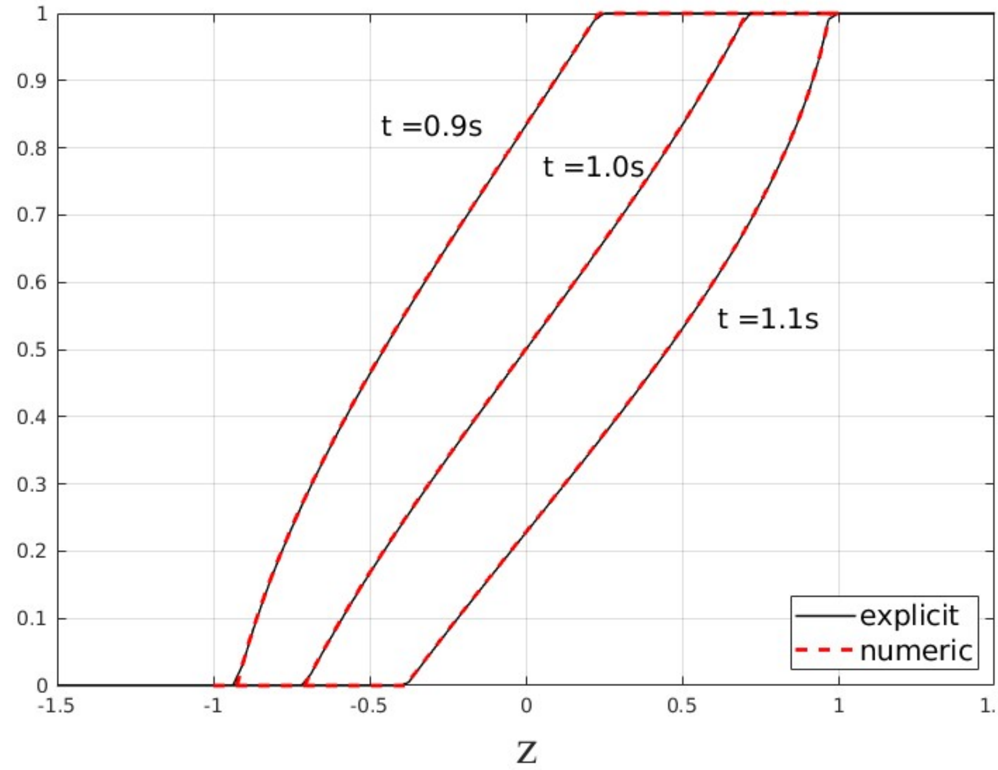

Figure 1: Cumulative distribution function at t=.9, t=1 and t=1.1 s; comparison between the explicit and numerical solutions (Example 1).

where ρ_L and ρ_R are the densities at the left and right extremes of the lane, respectively, and L is a positive parameter that measures how quickly the density of the cars changes from ρ_L to ρ_R (for $\rho_L \neq \rho_R$, as L tends to zero, ρ_0 becomes discontinuous at $x = 0$).

When $\rho_L < \rho_R$ (i.e. when for $t = 0$ the density of the cars increases in the direction of travel) there is a critical time T_c beyond which the propagation of the kinematic wave is no longer described by equation (19) because a shock wave is formed. Under our hypotheses, this happens at the time

$$T_c = \frac{2L\rho_m}{v_0(\rho_R - \rho_L)}. \tag{22}$$

Let us assume that v_0 and ρ_m are assigned quantities while $(\rho_R - \rho_S)$ and L are expressed by the two independent positive-valued random variables Θ_1 and Θ_2, respectively. Precisely, assume that Θ_1 is uniformly distributed on the interval $[\alpha, \beta]$ and that Θ_2 has an exponential distribution of parameter λ, i.e. $p_{\Theta_2}(\theta_2) = \lambda e^{-\lambda\theta_2}\chi_{(0,\infty]}$, where $\chi_{(0,\infty]}$ is the indicator function of the interval $(0, \infty]$. Then, for $Z(\boldsymbol{\theta}) = \frac{v_0 T_c}{2\rho_m}$, taking into account that

$$\{Z \leq z\} = \{(\theta_1, \theta_2) \in \Lambda = [\alpha, \beta] \times (0, \infty) : z - \frac{\theta_2}{\theta_1} \geq 0\}, \tag{23}$$

from (7) and (22) we obtain the cumulative distribution function of Z

$$F_Z(z) = \int_{\{Z\leq z\}} p_{\boldsymbol{\Theta}}(\boldsymbol{\theta})d\boldsymbol{\theta} = \frac{1}{\beta - \alpha}\int_\alpha^\beta d\theta_1 \int_0^{\theta_1 z} \lambda e^{-\lambda\theta_2}\chi_{(-\infty,0]}d\theta_2 =$$

$$\begin{cases} 0 & \text{if} z \leq 0, \\ 1 - \frac{e^{-\lambda\alpha z} - e^{-\lambda\beta z}}{(\beta-\alpha)\lambda z} & \text{if} z > 0. \end{cases} \tag{24}$$

Figure 2 shows F_Z for $\alpha = 50$ and $\beta = 150$ cars per kilometer, and different values of λ.

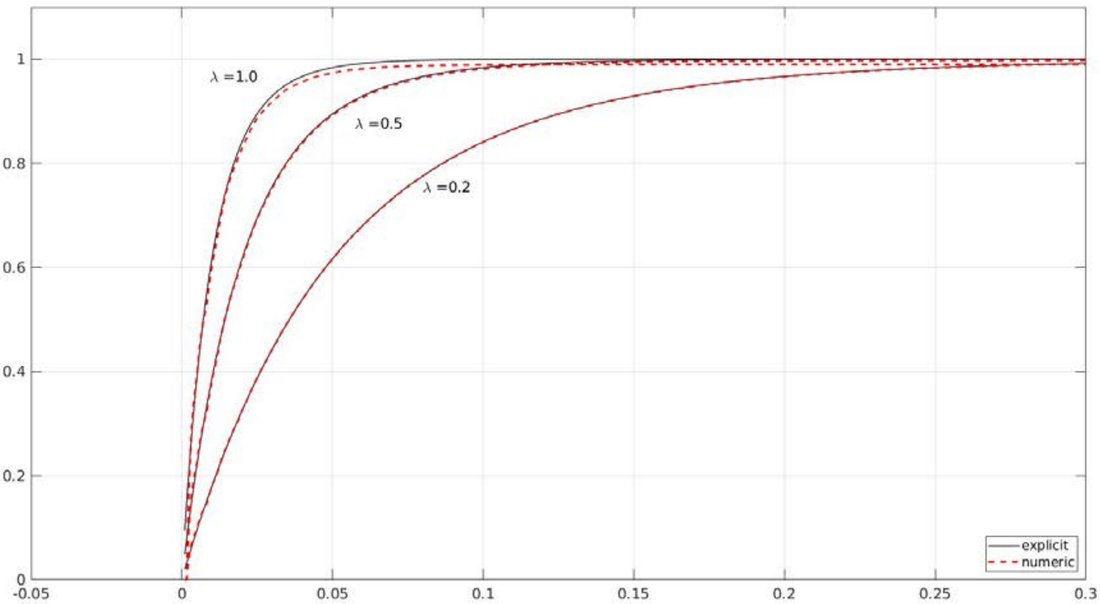

Figure 2: Cumulative distribution function for $\lambda = 1$, $\lambda = .5$ and $\lambda = .2$; comparison between the explicit and numerical solutions (Example 2).

Note that in a dynamic problem, the method presented in this article can be used to determine the probability that the variable Z exceeds a given limit, which in the context of structural reliability is known as *first-passage probability* [1]. Thus, e.g., if we want to know the probability that it turns out $T_c > 1/6h$ (ten minutes), i.e. that $Z > \frac{v_0}{12\rho_m}$, and we assume $v_0 = 130Km/h$ and $\rho_m = 450$ cars per kilometer, we obtain

$$P(\{T_c > 1/6\}) = \begin{cases} 1.87 \cdot 10^{-2} & \text{if} \quad \lambda = 1, \\ 1.14 \cdot 10^{-1} & \text{if} \quad \lambda = .5, \\ 3.98 \cdot 10^{-1} & \text{if} \quad \lambda = .2. \end{cases}$$

3.3 Masonry tower

We consider a masonry tower having height $H = 22m$ and a hollow rectangular cross-section of sizes $b = 9.5m$ and $h = 9m$, with corresponding thicknesses equal to $2.25m$ and $1.75m$ so that the total volume of the tower is $V = 1287m^3$. The tower is clamped at the base and constrained with lateral rods up to a height of about $9m$. The density of the masonry is $1900Kg/m^3$. The Young's modulus E is expressed by a random variable Θ_1 which has a lognormal distribution defined over the interval $[1.85, 2.65]GPa$

The maximum compressive strength σ_c has been chosen equal to $1.125 \cdot 10^{-3}$ times the average value of E and is equal to $2.5MPa$. The ductility in compression was assumed to be equal to 2. An input ground motion was applied to the tower for a time $T = 20s$. The accelerogram was recorded during the 2013 Fivizzano earthquake, having a PGA equal to $2.27m/s^2$. However, the PGA was chosen as the second random variable, uniformly distributed in the interval $[1.47, 3.07]m/s^2$ and denoted by Θ_2.

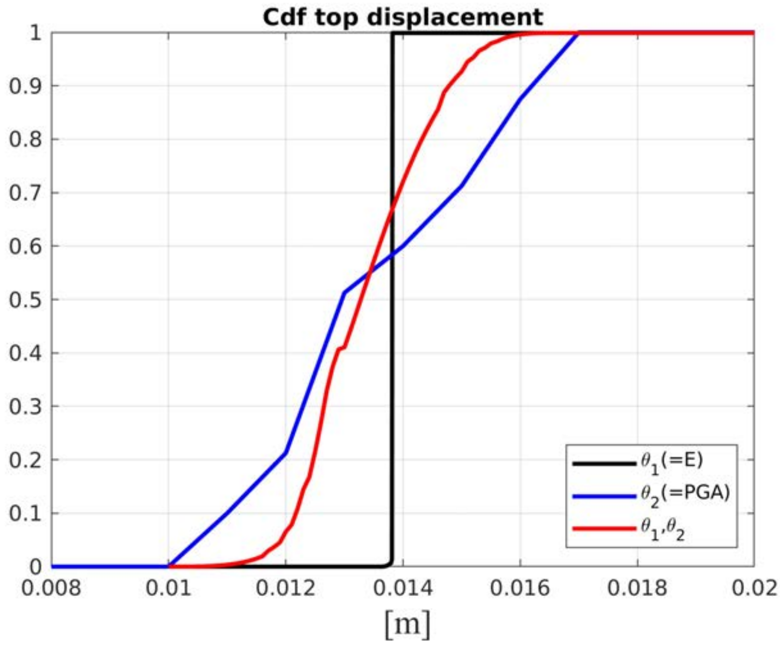

Figure 3: Cumulative distribution function of Z_1, when considering as random variables, respectively, Θ_1 (black), Θ_2 (blue) and both (red).

The dynamic behavior of the tower was analyzed with the MADY code [2]. Denoted by u the displacement at the top and by V_c the cracked volume of the tower, the cumulative distribution functions of the two variables

$$Z_1 = max\{|u(t)| : t \in [0, T]\} \tag{25}$$

and

$$Z_2 = max\{V_c(t)/V : t \in [0, T]\} \tag{26}$$

were determined.

Figures 3 and 4 show the CDF of the variable Z_1 and Z_2, respectively. The black, blue, and red curves are obtained considering as random variables, respectively, only $\Theta_1 = E$, only $\Theta_2 = PGA$, and both. In figure 3, it can be observed that the uncertainty of Young's modulus has little effects on the behavior of Z_1, which has a probability concentrated entirely within a small neighborhood of displacement $u = 11.6mm$. Conversely, as the PGA varies, the probability is distributed fairly uniformly within the interval of $[10, 15]mm$.

Similarly, in the case of the cracked volume (Figure 4), the influence of the PGA uncertainty is greater than that of Young's modulus, but the difference is less relevant then that found in the case of the tower's top displacement.

4 CONCLUSIONS

The examples presented in this work demonstrate that the proposed method is easy to apply even when there are several random variables among the input parameters. The method can

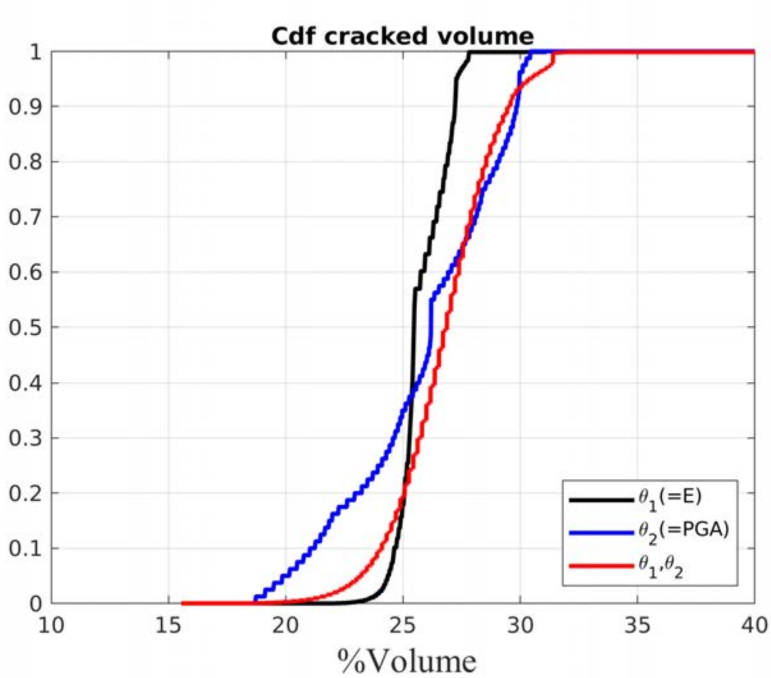

Figure 4: Cumulative distribution function of Z_2, when considering as random variables, respectively, Θ_1 (black), Θ_2 (blue) and both (red).

also be used to determine the *first passage probability*, i.e. the probability that certain quantities exceed a given limit, a topic of interest in determining the dynamic reliability of structures. Furthermore, although in the present paper only independent output variables have been considered (i.e., the cracked volume and the top displacement in the third example), the method can also be used to determine the (joint) cumulative distribution function of *vector-valued* random variables.

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