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CORRELATED GAUSSIAN PROCESS LATENT FORCE MODELS FOR RECOVERING MULTIPLE FORCES

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Abstract. *In engineering, there is much interest in quantifying the forcing that operational structures are subject to, either in real time or retrospectively. By having an understanding of the loading history, one is better placed to make fatigue damage accrual predictions, which are essential for estimating the remaining useful life of engineering assets. Although direct measures of the forcing experienced by a structure are challenging to acquire, there are low cost and accessible sensing solutions for obtaining the response of the structure such as accelerometers. Through the use of Gaussian process latent force models, it is then possible to probabilistically recover the latent states of the system from these measurements, which includes the loading history of the structure. Current solutions that take such an approach often assume that the input force acts as a single point load or that independent forces act on each mode. However, this view is somewhat limited given that the loads entering the system will be distributed over some part of the structure, introducing correlation between the forces.*

In this paper, the independence assumption is removed and the model transitions to a correlated forcing case, better reflecting the true external inputs acting on the structure. It is demonstrated that, beginning with a modal of a dynamic structure in the modal coordinate space, physical loads can be directly recovered in a Bayesian manner from a Gaussian process latent force model using only output data (accelerations). By explicitly modelling the dependency between each of the forces entering the system, the correlation that exists between each of the physical forces acting across all the modes is able to modelled.

Keywords: Correlated forces, Gaussian process latent force model, joint input-state estimation, Bayesian, force identification

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1 INTRODUCTION

During their operational lifespan, engineering assets are continuously subject to external forcing, either from environmental effects such as wind and wave loading, or due to working conditions such as in the rotational motion of bearing shafts. In many cases, additional challenges arise when these forces cannot be represented as a single point load. From a structural health monitoring perspective [1, 2], being able to monitor these applied loads will help ensure the safe and reliable operation of structures, with the eventual goal of quantifying remaining useful life in-situ [3]. However, obtaining direct loading measurements is very challenging, and often infeasible for large structures such as bridges and wind turbines [4]. Instead, more readily available dynamic quantities such as acceleration readings or strain measurements are often used to determine forcing values indirectly. Such an inverse problem can be formed as one of *joint input-state estimation*, where the states of the system are recovered in conjunction with the inputs recursively as a filtering problem [5–9].

More recently, the *Gaussian process latent force model* (GP-LFM) has been proposed as a solution to the joint input-state estimation problem, where the inputs to the system are treated as draws of a Gaussian process prior [10]. The latent force model was introduced by Alvarez et al. [11], where the mathematical model of a mechanistic system was embedded into a GP. Given that Gaussian process regression can be equivalently cast into a state space setting [12], the GP-LFM can be treated similarly [13], and has since resulted in its application across a range of input identification problems, including input estimation in a numerical model of a 76-storey building [14], joint input-state and physical parameter estimation of a three degree-of-freedom mechanical system [15] and wind forcing recovery in a suspension bridge [4]. Use of the GP-LFM in virtual sensing applications has also begun gathering momentum [16, 17].

Current solutions that take such an approach often assume that the system inputs acts as a single point load or that independent forces act on each mode. In this paper, we demonstrate how correlation between multiple forces acting on a system may be embedded into the latent force model. The proposed approach also enables direct recovery of the physical forces whilst constructing the GP-LFM in the modal space.

2 MECHANICAL SYSTEMS AS BAYESIAN STATE SPACE MODELS

At the core of the GP-LFM is the formation of a probabilistic state space model (SSM) of the system of interest. Broadly, SSMs provide a mathematical formulation for describing how systems with a set of inputs and outputs evolve in time, where the objective is to infer the hidden states of the system given observations of the system response. For instance, if tracking an object with only acceleration measurements, the internal states may be the displacement and velocity.

To construct a state space model relies on specifying both a *transition* function, f, and an *observation* function, h,

$$\mathbf{z}_t = f(\mathbf{z}_{t-1}, \mathbf{u}_t, q_t) \tag{1}$$

$$\mathbf{y}_t = h(\mathbf{z}_t, \mathbf{u}_t, r_t) \tag{2}$$

where f describes the evolution from one state ($\mathbf{z}_{t-1}$) to the next ($\mathbf{z}_t$) at time t, and h relates the state at the current time step ($\mathbf{z}_t$) to the observation ($\mathbf{y}_t$), both conditioned on an input $\mathbf{u}_t$. q_t is the noise on the process (on the model) and r_t is the noise from the observations, such as noisy sensor readings. From equation 1, it can be seen that the states evolve as a Markov process, with the current position of a hidden state depending only upon its previous position.

As the interest is in constructing a Bayesian interpretation of the state space model, f and h simply become densities instead of deterministic functions, leading to the following probabilistic model,

$$p(\mathbf{z}_t \mid \mathbf{z}_{t-1}, \mathbf{u}_t) = p(\mathbf{z}_t \mid f(\mathbf{z}_{t-1}, \mathbf{u}_t)) \tag{3}$$

$$p(\mathbf{y}_t \mid \mathbf{z}_t, \mathbf{u}_t) = p(\mathbf{y}_t \mid h(\mathbf{z}_t, \mathbf{u}_t)) \tag{4}$$

where equation 3 is the *transition model* and equation 4 is the *observation model*. The joint distribution of the latent states and observations can then be specified,

$$p(\mathbf{y}_{1:T}, \mathbf{z}_{1:T} \mid \mathbf{u}_{1:T}) = \left[p(\mathbf{z}_1 \mid \mathbf{u}_1) \prod_{t=2}^{T} p(\mathbf{z}_t \mid \mathbf{z}_{t-1}, \mathbf{u}_t) \right] \left[\prod_{t=1}^{T} p(\mathbf{y}_t \mid \mathbf{z}_t, \mathbf{u}_t) \right] \tag{5}$$

noting that the observations are conditionally independent to one another, given the hidden state. If one restricts the SSM to be linear in the transition and observation densities, along with a Gaussian likelihood (noise) model, then a specific case of the SSM is returned; the linear Gaussian state space model (LG-SSM). The corresponding transition and observation models can be defined in continuous time as,

$$\dot{\mathbf{z}}(t) = F_s \mathbf{z}(t) + G_s \mathbf{u}(t) + \mathbf{q}(t) \qquad \mathbf{q}(t) = \mathcal{N}(0, Q) \tag{6}$$

$$\mathbf{y}_t = H \mathbf{z}(t) + D \mathbf{u}(t) + \mathbf{v}(t) \qquad \mathbf{v}(t) = \mathcal{N}(0, R) \tag{7}$$

where F_s is the transition matrix, G_s the input matrix, H the observation matrix, and D the feed-through matrix, which may be zero depending upon which states are observed. Equations 6 and 7 can be solved for exactly in a Bayesian framework with the Kalman filtering [18] and Rauch-Tung-Striebel (RTS) smoothing algorithms [19]. Though full mathematical definitions and implementation steps are omitted here, the reader can find them extensively covered in many textbooks, including [20, 21].

With the LG-SSM model specified, attention can now turn to how dynamic systems may be represented within such a formulation. The equations of motion for a general multi-degree-of-freedom (MDOF) linear dynamic system and be represented by the following second order ordinary differential equation,

$$M\ddot{\mathbf{x}} + C\dot{\mathbf{x}} + K\mathbf{x} = \mathbf{u} \tag{8}$$

where M, C, K are mass, damping and stiffness matrices related to each degree-of-freedom. To avoid having to specify a predetermined number of degrees-of-freedom to capture the behaviour of the system, it may be more appropriate to shift from the physical to the modal space, where one only needs to truncate the number of overall modes to be considered. Defining Ω and Z as diagonal matrices of natural frequencies and damping ratios respectively, the equivalent governing equation can be transformed to,

$$\ddot{\mathbf{z}} + 2Z\Omega\dot{\mathbf{z}} + \Omega^2\mathbf{z} = \Phi^T\mathbf{u} \tag{9}$$

where $\mathbf{z}$ is a projection of the physical states, $\mathbf{x}$, onto the mass normalised mode vectors, Φ.

$$\mathbf{z} = \Phi \mathbf{x} \tag{10}$$

By constructing the inputs (RHS) as shown in equation 9, it is possible to represent the system forcing as a projection of the physical forces onto the relevant modeshape values. This is crucial to allowing the GP-LFM to directly recover the physical forces acting on the system, as will be discussed further in the proceeding section.

In the transformed coordinate space, each mode is decoupled, and so the rth mode can be represented independently as a system of first order differential equations,

$$\dot{z}_r = z_r \tag{11}$$

$$\ddot{z}_r = -w_r^2 z_r - 2Z_r w_r \ddot{z}_r + \tilde{\Phi}_r \mathbf{u} \tag{12}$$

allowing conversion to an LG-SSM of the form,

$$\dot{\mathbf{z}} = \begin{bmatrix} \mathbf{0} & \mathbb{I} \\ -\Omega^2 & -2Z\Omega \end{bmatrix} \mathbf{z} + \begin{bmatrix} \mathbf{0} \\ \tilde{\Phi} \end{bmatrix} \begin{bmatrix} \mathbf{u}_1 \\ \vdots \\ \mathbf{u}_{Np} \end{bmatrix} \tag{13}$$

for Np external inputs. Note that $\tilde{\Phi}$ indicates the evaluation of the modeshapes at the relevant forcing locations, with subscript r relating to a particular mode.

3 GP-LFMS FOR CORRELATED INPUTS

Now that the mathematical machinery for constructing mechanical systems as LG-SSMs has been covered, discussion regarding how these types of models may be built into a GP-LFM can be considered.

3.1 GP-LFMs: A brief overview

To transition from the modal space SSM in Section 2 to a latent force formulation, the core operation is to augment the internal states $\mathbf{z}$ with the inputs $\mathbf{u}$ such that the GP-LFM states are given by,

$$\mathbf{q} = [z_1, z_2, ..., z_R, \dot{z}_1, \dot{z}_2, ..., u_1, u_2, ..., u_R, \dot{u}_1, \dot{u}_2, ...\dot{u}_R] \tag{14}$$

The corresponding LFM transition model is then formed as,

$$\dot{\mathbf{q}} = \underbrace{\begin{bmatrix} F_s & G_s \\ \mathbf{0} & F_k \end{bmatrix}}_{K} \mathbf{q} + Lw \tag{15}$$

At this point, it will be useful to break down each component of the transition model:

F_s - This matrix represents the F_s matrix of the standard SSM defined in the previous section, and therefore encapsulates the transition behaviour of the internal dynamic states.

G_s - This component is the G_s matrix seen in the previous section, and governs how the physical forces enter the system as latent forces. This block of the F matrix also defines how dependencies between forces enter the LFM, and is therefore how correlation between different forces acting on the system is built into the LFM.

$\mathbf{0}$ - Matrix of 0s.

F_k - Transition matrix of the companion form of the stochastic differential equation (SDE) representation of a Gaussian process covariance function, and so is governed by the choice of kernel. This part of the LFM governs the covariance (which very loosely, can be thought of as the possible shape of the forcing signals) of the GP that models the latent forces.

L - Matrix of 0s other than at positions relating to the derivates of the latent forces.

w - White noise process with density, $S(w)$, corresponding to the chosen covariance function.

With F_s and G_s already defined, F_k and w remain to be specified. As discussed above, both of these terms are related to the choice of covariance function in the SDE representation of the Gaussian process prior. In this work, a Matérn kernel with smoothness parameter set to $3/2$, with a separate kernel for each individual force. The state space model for a Matérn 3/2 is defined as,

$$\dot{\mathbf{k}} = \begin{bmatrix} 0 & 1 \\ -\lambda^2 & -2\lambda \end{bmatrix} \mathbf{k} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w(t) \tag{16}$$

with a spectral density of w,

$$S(w) = \frac{2\sigma_f^2 \pi^{1/2} \lambda^3 \Gamma(2)}{\Gamma(3/2)} \tag{17}$$

One will note that a small number of *hyperparameters* of the GP-LFM now arise, defined as $\boldsymbol{\theta} = [\sigma_f, \lambda]$. $\lambda = \sqrt{(}3)/l$, where l is the characteristic lengthscale and σ_f the noise variance parameter, which represent the hyperparameters of a standard GP kernel [10]. As in the standard approach to Gaussian process regression, these hyperparameters may be optimised w.r.t some loss criterion. Within a filtering recursion, the energy function, $\phi(\boldsymbol{\theta})$, that is computed at each filter pass-through serves as a sensible candidate to minimise given that it is proportional to the negative log marginal likelihood of the observations [20],

$$\phi(\boldsymbol{\theta}) \propto -p(\mathbf{y}_{1:T} \mid \boldsymbol{\theta}) \tag{18}$$

In this work, optimisation is performed using stochastic gradient descent, with the relevant gradients of equation 18 assessed using the auto differentiation library JAX [22].

The final step before the GP-LFM may be implemented is to discretise the system such that the filtering and smoothing may be computed in a finite number of operations. A convenient approach here is to first construct an intermediate variable, Ψ,

$$\Psi = \Delta t \exp_m \begin{bmatrix} F & LS(w)L^T \\ \mathbf{0} & -F.T \end{bmatrix} \tag{19}$$

To obtain the discretised transition matrix, F_d, one can take the upper left quadrant of Ψ; $F_d = \Psi[:D, :D]$[1]. With regard to the process noise, Q_d, this corresponds to the upper right quadrant of Ψ, multiplied with ${F_d}^T$;which arises from the solution to the Lyapunov equation (more details can be found in Chapter 6.3 [23]); $Q_d = \Psi[:D, D:]{F_d}^T$.

[1]Here, the notation $[:D]$ implies from the first entry, in steps of 1, to the Dth -1 entry. Likewise, $[D:]$ is the index from the Dth entry to the final.

3.2 Embedding correlation between forces

As discussed above, by construction, the G_s matrix describes how the latent forces enter the GP-LFM, and therefore embedding correlation between forces is dictated by G_s.

One possible approach for capturing correlation across inputs, and the approach pursued here, is to model the latent forces directly as the physical forces, where G_s is populated with modeshape values corresponding to a particular modeshape and forcing location. That is, $\phi_{(r,j)}$ equals the modeshape for the rth mode at forcing location j. The full G_s matrix is written as,

$$G_s = \begin{bmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \\ \phi_{(1,1)} & \dots & \phi_{(1,Np)} \\ \vdots & \ddots & \vdots \\ \phi_{(R,1)} & \dots & \phi_{(R,Np)} \end{bmatrix} \tag{20}$$

where the system states corresponding to the latent forces are given as each of the physical forces acting on the system; $[\mathbf{u}_1, ..., \mathbf{u}_{Np}]$. Construction of G_s in such a manner allows the correlation between different modes and external forces to be explicitly captured, whilst enabling direct identification of the external inputs from the LFM. A second option for constructing G_s is rather than directly including the physical inputs acting on the system, is to recover latent forces for each mode. This second case will be left for future work.

4 RESULTS AND DISCUSSION

To investigate the proposed approach for embedding correlated forces into the GP-LFM, a case study consisting of a simulated fixed-free beam simultaneously forced by two external inputs is considered. To excite the system, one force is positioned at the half span of the beam, with the second located at the free end. Their time histories are presented in Figure 1. To model the response of the beam, the first five modes are considered, with the internal dynamic states of the system plotted in Figure 2.

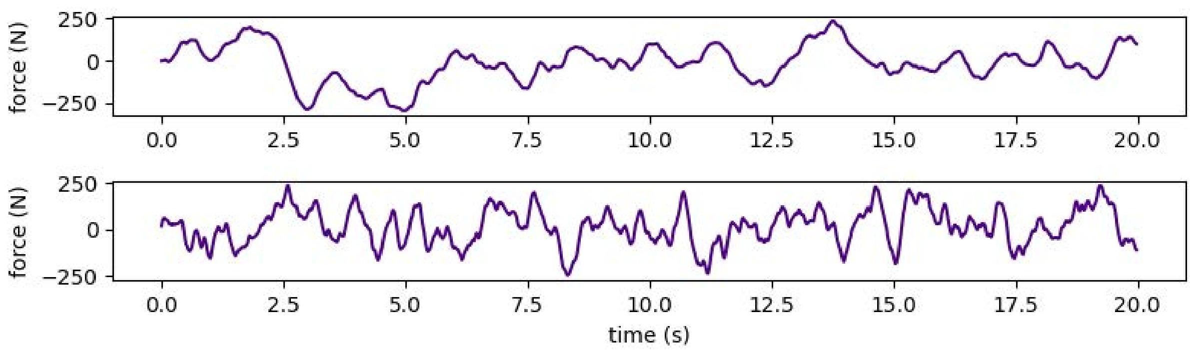

Figure 1: Time history of the two externals inputs.

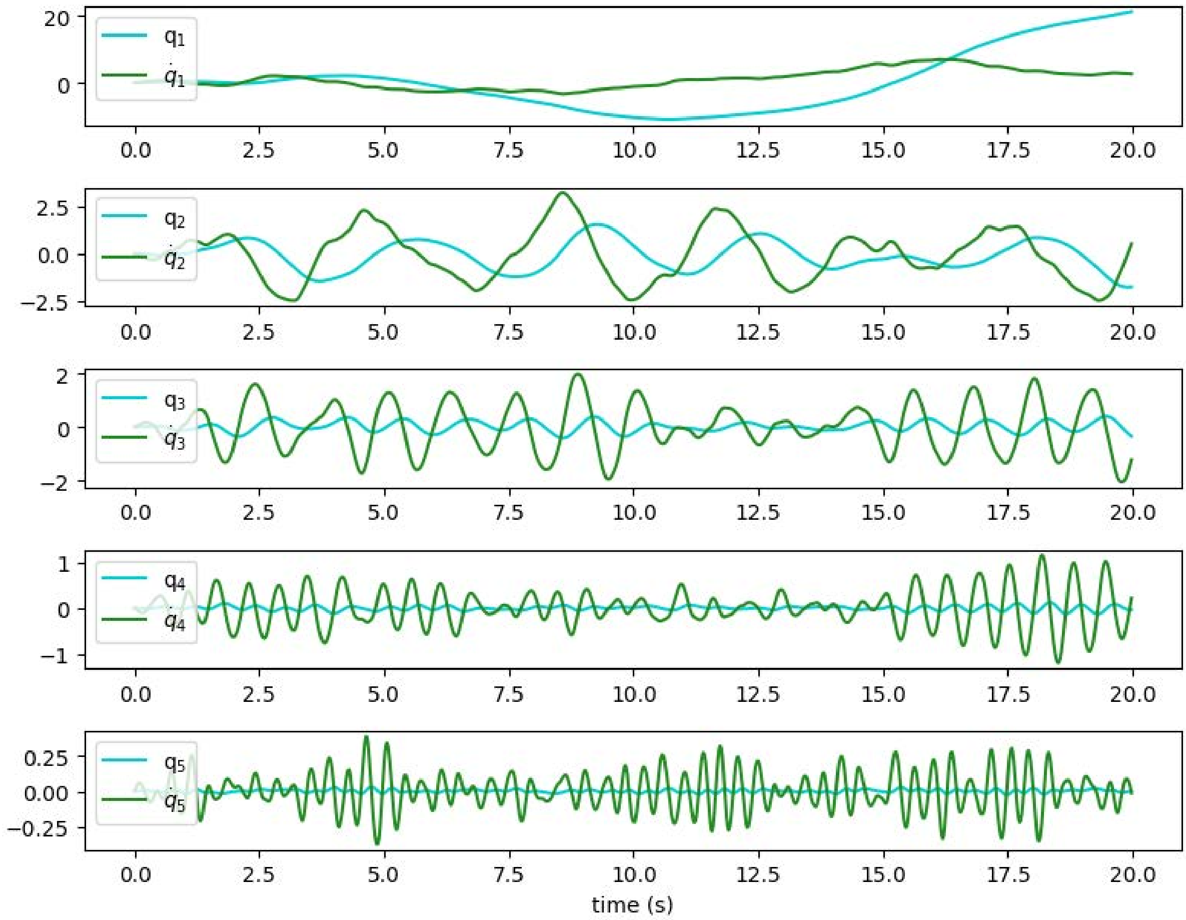

Figure 2: Hidden dynamic states of the first five modes of the system.

In terms of observing the system, it is assumed that there are five equally spaced accelerometers across the beam, with the first sensor placed at a quarter of the span from the root, and the final on the free end. To mimic more realistic operational conditions, the measurements are corrupted with a 1% [2] variance Gaussian white noise.

For the ith sensor, the corresponding observation model of the GP-LFM is written as,

$$\mathbf{y}_{t,i} = \begin{bmatrix} -\Omega^2 \mathbf{\Phi}_i & -2\zeta\Omega\mathbf{\Phi}_i \\ \mathbf{\Phi}_i \tilde{\Phi} & \mathbf{0} \end{bmatrix} \mathbf{q}_t + \mathbf{v}(t) \tag{21}$$

where $\mathbf{\Phi}_i$ are the R modeshapes at a particular sensing location.

In possession of an appropriate transition and observation model, inference over the physical forces can now commence. Running the filter-smoother with the simulated acceleration measurements returns the smoothed forcing predictions plotted in Figure 3. The results demonstrate that the input forces were able to be recovered across the time history of the simulated system. Where the expectation of the forcing prediction does not closely align with the true value, the bounds of the associated uncertainty returned by the predictions are able to enclose the actual forcing. This behaviour highlights the importance of the probabilistic approach taken, with the option to propagate this uncertainty forward into any subsequent assessment of the monitored structure.

[2]Relative to the variance of the true measurement.

Comparing the accuracy of the forcing identification between the two forces, it can be seen that, generally, the second force is more successfully recovered, with tighter uncertainty margins. Given that the second force represents the forcing applied at the free end of the beam, this forcing introduces a greater degree of energy into the system, and as such, is more easily recoverable from the observed responses.

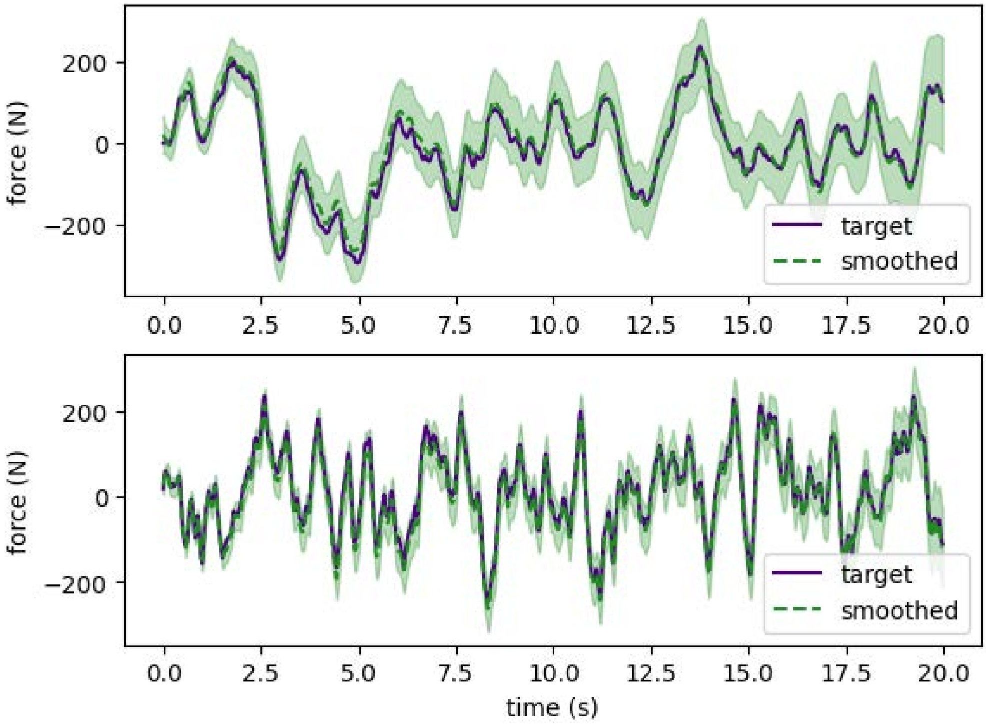

Figure 3: Smoothing predictions returned by GP-LFM. Shaded area represents 3 standard deviations of the predictive distribution.

5 CONCLUSIONS

In this paper, it has been shown that the time history of multiple external loads acting on a system may be jointly recovered from output only measurements. By projecting the forces onto the relevant modeshapes, direct estimates of the physical forces are returned by a Gaussian process latent force model, with the correlation between the individual modes of the dynamic system and external inputs included directly in the model, more appropriately representing the relationship of the forces acting across the modes than in an equivalent independent model.

In future work, the approach introduced here will be compared to recovering modal forces, where rather than the latent forces representing the individual physical forces that are acting on the system, a contribution from all the external forces related to each mode are the latent forcing states. Particular attention will be given to how correlation may also be embedded in the modal forcing setting, as well as how progress can be made towards recovering distributed loads.

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