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A REDUCED ORDER MODEL APPROACH FOR FUZZY FIELDS ANALYSIS

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Abstract. *Due to imprecision, scarcity, and spatial variation in input data, the response of systems with governing parameters that possess spatial dependencies is hard to characterize accurately. Under such a scenario, fuzzy fields become an efficient tool for solving problems that exhibit uncertainty with a spatial component. Nevertheless, the propagation of the uncertainty associated with input parameters characterized as fuzzy fields towards the output response of a model can be quite demanding from a numerical viewpoint. Therefore, this contribution proposes an efficient numerical strategy for forward uncertainty quantification under the presence of fuzzy fields. In order to decrease numerical costs associated with uncertainty propagation, full system analyses are replaced by a reduced order model. This reduced order model projects the equilibrium equations to a small-dimensional space, which is constructed using a single analysis of the system plus a sensitivity analysis. The associated basis is enriched to ensure the quality of the approximate response and numerical cost reduction. A case study of seepage analysis shows that with the presented strategy, it is possible to accurately estimate the fuzzy total flow, with reduced numerical efforts.*

Keywords: Fuzzy fields, Reduced order model, Seepage analysis, Permeability, Spatial uncertainty

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1 INTRODUCTION

Seepage analysis is of utmost importance in several practical engineering problems such as dam design [1]. In this context, seepage can be quantified by resorting to, e.g., finite elements. However, the parameters that govern the seepage phenomenon, such as permeability, can be seldom quantified precisely [2, 3]. Several difficulties are encountered in practice when characterizing permeability, such as imprecision and scarcity in field measurements, anisotropy in vertical and horizontal directions, and spatial variation [4]. Under such a scenario, fuzzy fields appear as a viable tool for treating problems that exhibit imprecision with a spatial component [5]. Nonetheless, dealing with fuzzy fields imposes a major challenge, as it becomes necessary to propagate uncertainty considering spatial dependencies, which can be quite challenging when large-scale numerical models are involved [6]. This difficulty stems directly from (1) the by-definition orthogonality between any two intervals in a set of intervals; (2) the computational burden associated with the multiple calls to the model.

This contribution proposes an approach for seepage analysis where uncertainty associated with permeability is characterized by means of fuzzy fields. In this context, fuzzy fields are defined as a natural extension of the Inverse Distance Weighting framework that is commonly applied in an interval field context (see Faes and Moens [7]). Concerning the propagation of the fuzzy field, the traditional $\alpha-$level optimization strategy is adopted for calculating the membership function associated with seepage responses in a discrete manner [8]. Within this optimization process and to decrease numerical costs, full finite element analyses are replaced by a reduced order model that projects the system's equations to a small-dimensional space. The basis associated with the reduced order model is constructed by means of a single analysis of the system plus a sensitivity analysis [9]. This reduced basis is enriched adaptively as the $\alpha-$level optimization strategy progresses to protect the quality of the approximations provided by the reduced order model. A numerical example illustrates that the proposed strategy allows for characterizing the fuzzy seepage flow, with improved numerical efficiency.

2 FORMULATION OF THE PROBLEM

2.1 Finite element formulation

The partial differential equation that governs the 2D steady-state confined seepage problem [10] under uncertainty is:

$$\frac{\partial}{\partial x}\left(k_H\left(\xi\right)\frac{\partial h}{\partial x}\right)+\frac{\partial}{\partial y}\left(k_V\left(\xi\right)\frac{\partial h}{\partial y}\right)=0 \tag{1}$$

where ξ represents uncertainty in soil permeability; k_H and k_V correspond to the horizontal and vertical permeabilities, respectively; and h is the hydraulic head. x and y are the Cartesian coordinates of the domain Ω.

Equation 1 is solved by means of the finite element (FE) method,

$$\boldsymbol{K}\left(\xi\right)\boldsymbol{h}\left(\xi\right)=\boldsymbol{q}\left(\xi\right) \tag{2}$$

where $\boldsymbol{K}\left(\xi\right)$ is the matrix associated with soil permeabilities; $\boldsymbol{q}\left(\xi\right)$ is the vector representing nodal flow; and $\boldsymbol{h}\left(\xi\right)$ is the vector that describes the system's response, that is, hydraulic head. As noted from Equation 2, the uncertainty affecting the system's matrix $\boldsymbol{K}\left(\xi\right)$ and the flow $\boldsymbol{q}\left(\xi\right)$ propagates to the hydraulic head $\boldsymbol{h}\left(\xi\right)$.

2.2 Fuzzy Fields

A possible way of quantifying the uncertainty associated with the system's response corresponds to applying techniques of fuzzy analysis [11]. Thus, the uncertain parameter ξ can be characterized as a fuzzy variable $\hat{\xi}$. In this context, a fuzzy variable can be interpreted as a collection of intervals for different membership levels, where these intervals are indexed by a membership function $\mu_{\hat{\xi}}(\xi) \in [0, 1]$. Consequently, as the input parameters are characterized as fuzzy variables, the response will depend on the membership level under analysis. This implies that the system's response is a fuzzy variable as well.

One approach to determine the membership function of the response is to use the previous interpretation of a fuzzy variable. For this purpose, the membership function $\mu_{\hat{\xi}}(\xi)$ associated with the fuzzy variable $\hat{\xi}$ is analyzed for discrete membership values α_l, with $l = 1, \ldots, n_c$, where n_c indicates the number of discrete levels considered. In other words, for each of these membership levels α_l, there will be an interval associated with the uncertain variable ξ, that is:

$$\xi^I_{\alpha_l} = \left\{ \xi \in \Xi : \mu_{\hat{\xi}}(\xi) \geq \alpha_l \right\}, \ \alpha_l \in (0, 1], \ l = 1, \ldots, n_c \tag{3}$$

where $\xi^I_{\alpha_l}$ represents the possible set of values that ξ can assume for an α_l-cut of the membership function; and Ξ is the set that contains all physical values that ξ can accept. Note that this level corresponds to an interval whose lower and upper limits are $\underline{\xi}_{\alpha_l}$ and $\bar{\xi}_{\alpha_l}$, respectively. For a given membership level α_l, the response of interest r will be contained in an interval $r^I_{\alpha_l}$ with lower $\underline{r}_{\alpha_l}$ and upper $\bar{r}_{\alpha_l}$ bounds.

In mathematical terms,

$$\underline{r}_{\alpha_l} = \min_{\xi} \left(r(\xi) \right), \ l = 1, \ldots, n_c \tag{4}$$

$$\bar{r}_{\alpha_l} = \max_{\xi} \left(r(\xi) \right), \ l = 1, \ldots, n_c \tag{5}$$

In this contribution, the response of interest r can correspond to the seepage flow, where the lower $\underline{\xi}_{\alpha_l}$ and upper $\bar{\xi}_{\alpha_l}$ bounds, from equations 4 and 5, respectively; are determined by optimization. The characterization of input parameters by means of fuzzy variables, such as those described recently, assumes that parameters under study are only affected by uncertainty. Nevertheless, uncertainty is also dependent on spatial coordinates for some variables, such as soil properties. Therefore, in the presence of spatial dependencies, the fuzzy variable concept can be extended as a fuzzy field.

Consider the case presented in Figure 1, where a single input variable ξ exhibits spatial dependence. In that case, there is information (i.e., scarce physical measurements) of the input parameter at two specific locations on the domain Ω (x_1 and x_2 in Figure 1). These positions correspond to the control points, where the number of control points is represented by n_b. In Figure 1, $n_b = 2$. In each of these positions, it is possible to characterize the uncertain parameter as a fuzzy variable (represented by the green membership functions in Figure 1). For a specific membership level α, note that is possible to associate an interval, as in Equation 3, but now at each location. Using basis functions (see Faes and Moens [7]), the information in the control points is projected to any position on the domain Ω. As a result of that procedure, the interval field associated with α is obtained (red area in Figure 1). If this process is repeated for different membership levels, it discretely approximates the input parameters as a fuzzy field.

Note that: (1) this strategy is appropriate when there is limited data of an input parameter of the model (at specified locations on the domain), and (2) fuzzy fields allow propagating the uncertain input data to the finite element mesh.

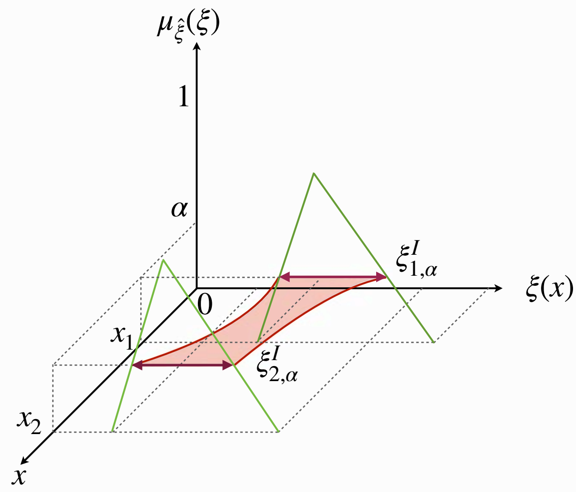


Figure 1: Fuzzy field schematic representation.

When an uncertain property is represented by a fuzzy field, it is necessary to determine how this property is discretely represented at the level of the individual finite elements in order to apply the finite element (FE) method. One way to perform the discretization of the fuzzy field corresponds to applying the midpoint method [12], which consists in assuming that such properties, within a given finite element, can be completely described by their value at the centroid of that finite element, whose coordinate is denoted by $\boldsymbol{x}_{C,q}$, with $q = 1, \ldots, n_e$ and n_e is the number of elements. Therefore, the information of the uncertain parameters at the n_b control points is propagated for each $\alpha-$level, for example, to a specific centroid $\boldsymbol{x}_{C,q}$ by means of:

$$\xi_{C,q} = \sum_{j=1}^{n_b} \psi_j(\boldsymbol{x}_{C,q}, \boldsymbol{X})\xi_j, \ \xi_j \in \xi^I_{j,\alpha} \tag{6}$$

where $\xi_{C,q}$ corresponds to the value of the uncertain parameter at the centroid of the element q, and $\psi_j(\boldsymbol{x}_{C,q}, \boldsymbol{X})$ are the basis functions evaluated at the centroid coordinates of the q-th finite element. $\boldsymbol{X}$ is a matrix that stores the control points coordinates, that is, $\boldsymbol{X} = [\boldsymbol{x}_1, \ldots, \boldsymbol{x}_{n_b}]$, and ξ_j denotes the value of the uncertain property at the j-th control point, which is contained in the interval $\xi^I_{j,\alpha}$ as shown in Figure 1.

From Equation 6, is clear that the uncertainty in the input parameters is reduced to the information contained in the control points. The basis functions ψ_j are defined in this work according to the following equation:

$$\psi_j(\boldsymbol{x}, \boldsymbol{X}) = \frac{w_j(\boldsymbol{x}, \boldsymbol{x}_j)}{\sum_{j_1=1}^{n_b} w_{j_1}(\boldsymbol{x}, \boldsymbol{x}_{j_1})}, \ j = 1, \ldots, n_b \tag{7}$$

where ψ_j is the j-th basis function with $j = 1, \ldots, n_b$; $w_j(\boldsymbol{x}, \boldsymbol{x}_j)$ is the weight function between a specific spacial coordinate $\boldsymbol{x}$ and the node location vector $\boldsymbol{x}_j$. In this proposal, the weight functions w_j correspond to the Inverse Distance Weighting function (see Faes and Moens [7]).

2.3 Reduced order model

Direct solution of equations 4 and 5 can be quite demanding from a numerical viewpoint, as they demand repeated evaluation of the equilibrium equation (see Equation 2). A possible means to decrease numerical costs consists of applying a reduced order model. Thus, the approximate response in terms of hydraulic head $\boldsymbol{h}^A(\xi)$, dependent on the considered fuzzy fields, can be expressed as the linear combination of several known components. These components constitute the reduced basis $\boldsymbol{\Phi}$, which is constructed by means of a single exact analysis of the system plus a sensitivity analysis of the response concerning the uncertain parameters [6]. This sensibility analysis demands performing a single system evaluation as well. As part of this sensibility analysis, the partial derivatives are calculated analytically using a direct method (see Haftka and Gürdal [9]) and evaluated at a nominal point ξ^0. The nominal point satisfies that $\mu_{\hat{\xi}}(\xi^0) = 1$.

The expression to obtain the approximate hydraulic head $\boldsymbol{h}^A(\xi)$ is given by:

$$\boldsymbol{h}(\xi) \approx \boldsymbol{h}^{\mathrm{A}}(\xi) = \boldsymbol{\Phi}\boldsymbol{\beta}(\xi) \tag{8}$$

where $\boldsymbol{\beta}(\xi)$ is a vector whose components depend on uncertain parameter. From Equation 2, which admits the characterization of the input parameters as fuzzy fields, and using the reduced basis $\boldsymbol{\Phi}$, the reduced system corresponds to:

$$\boldsymbol{K}_R(\xi)\boldsymbol{\beta}(\xi) = \boldsymbol{q}_R(\xi) \tag{9}$$

where $\boldsymbol{K}_R(\xi)$ is the stiffness matrix of the reduced system: $\boldsymbol{K}_R(\xi) = \boldsymbol{\Phi}^T\boldsymbol{K}\,(\xi)\,\boldsymbol{\Phi}$, and $\boldsymbol{q}_R(\xi)$ is the reduced flow vector: $\boldsymbol{q}_R(\xi) = \boldsymbol{\Phi}^T\boldsymbol{q}\,(\xi)$.

To control the error introduced by the reduced order model, one can investigate the residual error associated with the equilibrium equations considering the approximate response (see Gogu et al. [13]). That is,

$$\varepsilon(\xi) = \frac{\|\boldsymbol{K}(\xi)\boldsymbol{h}^A(\xi) - \boldsymbol{q}^A(\xi)\|}{\|\boldsymbol{q}^A(\xi)\|} \tag{10}$$

where $\varepsilon(\xi)$ is the error measure and $\|\cdot\|$ denotes Euclidean norm. The error $\varepsilon(\xi)$ is monitored at each $\alpha-$cut during the optimization process (i.e. solution of equations 4 and 5). The reduced basis was updated each time the largest error produced for one limit (for a specific membership level) exceeded a predefined threshold ε_t.

3 EXAMPLE

The analysis of steady-state confined seepage below an impermeable dam is considered to illustrate the proposed approach. The geometrical definition of the system is based on [14]. The dam is founded on a permeable soil layer limited by an impermeable rock layer. The objective is to determine the flow that drains downstream of the dam. The dam's upstream side retains a water column of a height of 10 [m]. The soil layer has a depth of 20 [m], and its horizontal k_H and vertical k_V permeability are characterized as fuzzy fields. Four control points were considered, where the dependence between both permeabilities is included considering $0.1k_H \leq k_V \leq k_H$ [15, 16]. The membership functions associated with each control point are shown in Figure 2.

A simple finite element model is considered, which comprises 3183 nodes and 1498 quadratic triangular elements. The system was studied considering two models: the exact model and the approximate model $R1$. The results were evaluated considering that (a) there is no basis updating process ($\varepsilon_t \to \infty$), and (b) there is a basis updating process with a threshold of $\varepsilon_t = 10^{-4}$.

Figure 3 presents the estimation of the membership function associated with the flow response. The results produced with the reduced basis R1 provide a satisfactory match with the exact system's response. In case (a), some minimum discrepancies can be noted for low values of the membership function on the left side. These differences are not present when the procedure considers the basis updating strategy. Note that the proposed approach brings good benefits regarding the computation time. The speedup factor associated with R1 is 43.8 for $\varepsilon_t \to \infty$ while the speedup factor associated with R1 for $\varepsilon_t = 10^{-4}$ is 32.9. Concerning the basis updating process, to maintain the error under the established threshold, a total of 3 exact additional analyses were required.

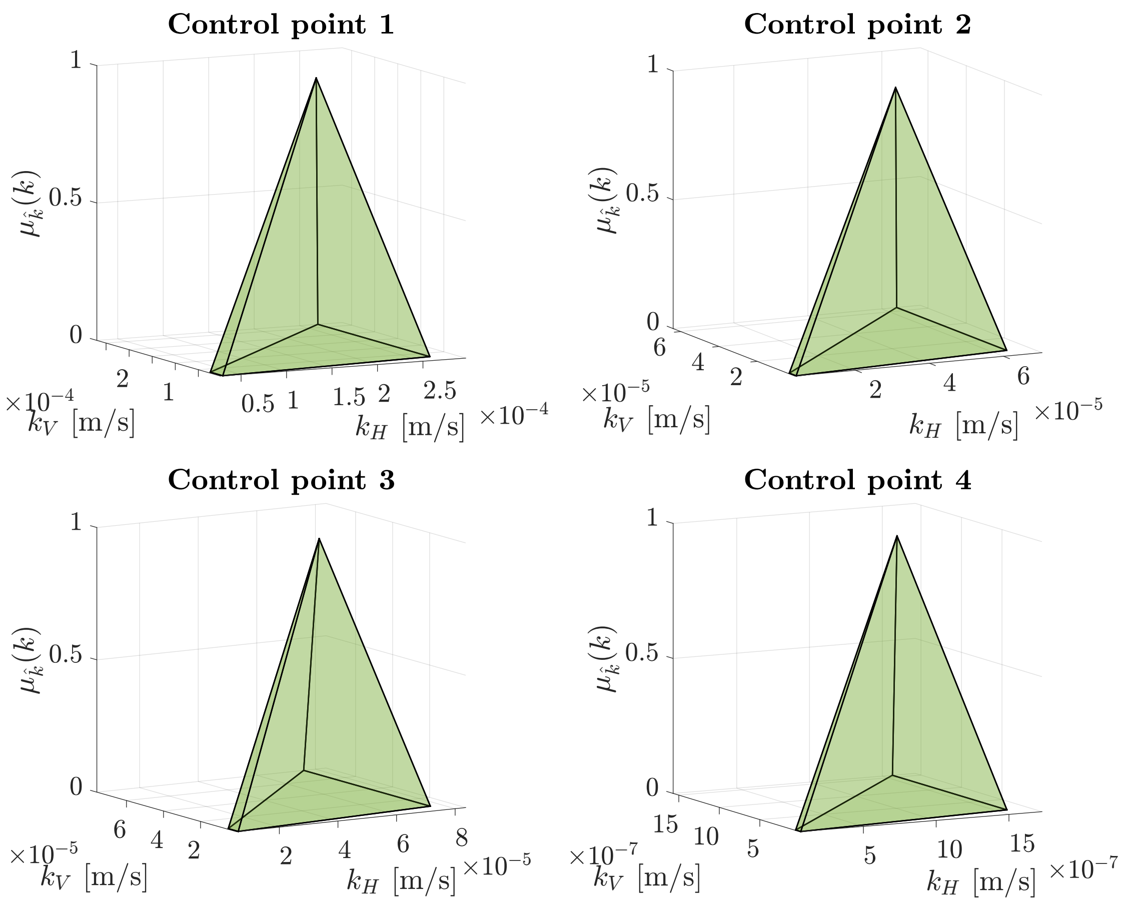
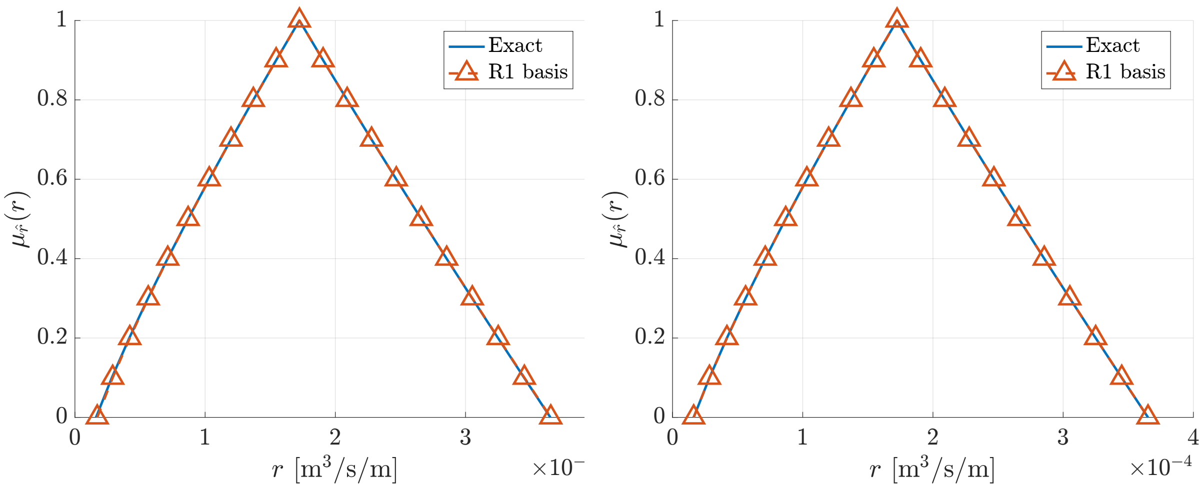


Figure 2: Membership function of permeabilities at control points. Each control point is located at: (a) $x = 30$ [m], $y = 15$ [m]. (b) $x = 30$ [m], $y = 4.5$ [m]. (c) $x = 75$ [m], $y = 9$ [m]. (d) $x = 75$ [m], $y = 2$ [m].

4 Conclusions

This contribution presents a technique to estimate the fuzzy response of a seepage problem considering the spatial uncertainty in soil permeability by applying a reduced order model. The approach is formulated to propagate the uncertain permeability characterized by fuzzy fields through an optimization scheme. In particular, the subsequent challenges are addressed: (1)



(a) (b)

Figure 3: (a) Membership function associated with the flow response for $\varepsilon_t = \infty$. (b) Membership function associated with the flow response for $\varepsilon_t = 10^{-4}$.

spatial dependencies in the horizontal and vertical permeability; and (2) the numerical cost associated with the resolution of the exact system. The results demonstrate that a precise estimation of the fuzzy responses can be obtained at reduced numerical efforts, controlling the quality of the results. Furthermore, it exhibits that fuzzy fields are a useful strategy for spatial uncertainty quantification under limited data. Nevertheless, the exhibited results should be regarded as an initial approximation of fuzzy field analysis. Forthcoming studies steps will aspire to explore more complex systems, for example, considering other types of responses and extensions to more physical dimensions.

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