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ON PROCESSING HETEROGENEOUS SOURCES OF LIMITED DATA FOR UNCERTAINTY QUANTIFICATION IN A POSSIBILISTIC FRAMEWORK

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Abstract. *This contribution aims at highlighting the use of possibility theory in uncertainty quantification for engineering problems by inferring distributions from different sources of limited data and combining the gained knowledge to serve as a basis for subsequent decisions. The mathematical framework of possibility theory offers the ability to infer information in the form of concrete mathematical expressions from data in the narrow sense, while also proving convenient for the simultaneous or subsequent incorporation of other sources of information through transformation, conjunction and propagation of possibilistic distributions, while continuously resting on a solid mathematical foundation that guarantees the conservative validity of the result.*

In a robot localization scenario carried out in a lab environment, only a few noisy measurements from a UWB (ultra-wideband) sensor setup as well as some additional knowledge such as room dimensions and sensor characteristics are available. The impact of progressively adding and combining information from these data sources is analyzed, showcasing the advantages and opportunities of possibility theory for processing heterogeneous sources of limited data for uncertainty quantification, while also discussing practical limitations of the current state of the framework.

Keywords: Uncertainty Quantification, Possibility Theory, Limited Data, Inference, Mobile Robot Localization, Indoor Positioning

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1 INTRODUCTION

Uncertainty quantification deals with the appropriate description, propagation and analysis of aleatory, i.e. random and unavoidable, uncertainties as well as epistemic uncertainties, which reflect a lack of available knowledge. Given enough data, aleatory uncertainties are often perfectly quantifiable by the use of probabilistic distributions. However, a challenge engineers routinely face is the emergence of epistemic uncertainty due to a lack of knowledge in the form of limited data. The simultaneous presence of both types of uncertainty, i.e. the existence of polymorphic uncertainty, motivates the use of imprecise probabilities as one approach to its adequate quantification.

Data or, more generally, information sources, can include not only data points in the classical sense of measured observations, but also information of other types. This could comprise physical boundary conditions, precisely known probability distributions, proven heuristics, expert opinion, educated guesses, and others. The challenge then lies in the appropriate combination of these different sources of information.

Indoor localization is a widely studied problem in literature, where a wide variety of different approaches and methods for state estimation find their application [1]. UWB (ultra-wideband) sensors are a readily used technology in this context, as GPS is usually unavailable in indoor environments and Wi-Fi often cannot meet required accuracy needs [2].

A novelty of this contribution lies in the application of possibility theory to such UWB positioning problems by presenting a first practical application of the methods developed in [3]. The contents of this paper are laid out as follows: After a coarse overview of relevant methods from possibility theory, mainly following the theory from [3], an experimental workflow for a static robot localization problem is introduced step by step. The influence of different data sources is studied, visualized and discussed.

2 POSSIBILISTIC DESCRIPTION OF UNCERTAINTY

The elementary possibility function $\pi_{\widetilde{X}}(x)$ of the uncertain variable $\widetilde{X}$ is formally equal in its definition to a fuzzy membership function, where it maps the sample space Ω to the interval $[0, 1]$. A higher value denotes a higher possibility of occurrence. It is the basic element of possibility theory, capable of describing imprecise probabilities, as it induces a measure of possibility

$$\Pi_{\widetilde{X}}(x) = \sup_{\xi < x} \pi_{\widetilde{X}}(\xi) \tag{1}$$

as well as a measure of necessity

$$N_{\widetilde{X}}(x) = \inf_{\xi > x} (1 - \pi_{\widetilde{X}}(\xi)), \tag{2}$$

which, in combination, bound a so-called credal set of cumulative probability distributions [4]. That means, any valid cumulative probability distribution describing $\widetilde{X}$ is bounded by the possibility Π from above and by the necessity N from below, adhering to the concept of consistency [5].

Take, for example, the interval-valued parameter $x \in \mathbb{R}$ that is known to be included in the closed interval $[a, b]$, where $a, b \in \mathbb{R}$ and $a \leq b$. It can be modeled possibilistically using the quasi-vacuous possibility function $\pi_{\widetilde{X}}(x)$ shown in Figure 1a. As no further information on the parameter is available, any probability function assigning all of its mass within the interval $[a, b]$ is a valid cumulative probability distribution for $\widetilde{X}$, as shown in Figure 1b.

Hose [3] distinguishes between three types of quantitative possibility functions. The *descriptive distribution* π describes possible distributions of an uncertain variable. It is used to model imprecise input variables. The *confidence distribution* γ describes a precise but unknown parameter. For example, the data-based identification of a parameter distribution that defines a parameter-dependent statistical model results in a confidence distribution. Finally, there is the *prediction distribution* κ, which describes the distribution of the next data point based on a set of available data. All three types of possibility functions follow the same mathematical principles and can be handled by the same tools. A notable exception are the two latter distributions for which subnormality, i.e. a maximal possibility of less than one, is permitted. See [3] for a detailed discussion thereof. For the sake of notational and conceptual simplicity, all three types of possibility distributions will in the following be described by the letter π.

2.1 Propagation of possibilistic uncertainty

The propagation of a possibilistically described uncertain variable is performed by the use of a special form of the extension principle [6]

$$\pi_{\widetilde{Y}}(y) = \sup_{y \in f(x)} \pi_{\widetilde{X}}(x), \tag{3}$$

where the function f is the mapping of the uncertain input variable $\widetilde{X}$ to the uncertain output variable $\widetilde{Y}$. Whenever $\widetilde{X}$ is multidimensional and only marginal distributions are available, a valid joint distribution that preserves consistency must first be found. Its construction varies according to the knowledge about the dependence of the marginals $\widetilde{X}_i$. In the case of unknown interaction for example, a valid joint distribution is given by

$$\pi_{\widetilde{X}}(x) = \mathcal{J}^{\mathrm{UI}}\left(\pi_{\widetilde{X}_1}(x_1), \ldots, \pi_{\widetilde{X}_{m(x_m)}}\right) \tag{4}$$

with the copula for unknown interaction

$$\mathcal{J}^{\mathrm{UI}}(\pi_1, \ldots, \pi_m) = \min\left(1, m \cdot \min_{i=1,\ldots,m} \pi_i\right). \tag{5}$$

For a detailed explanation of different types of dependence and the corresponding constructions of valid joint distributions refer to [3, p. 78ff].

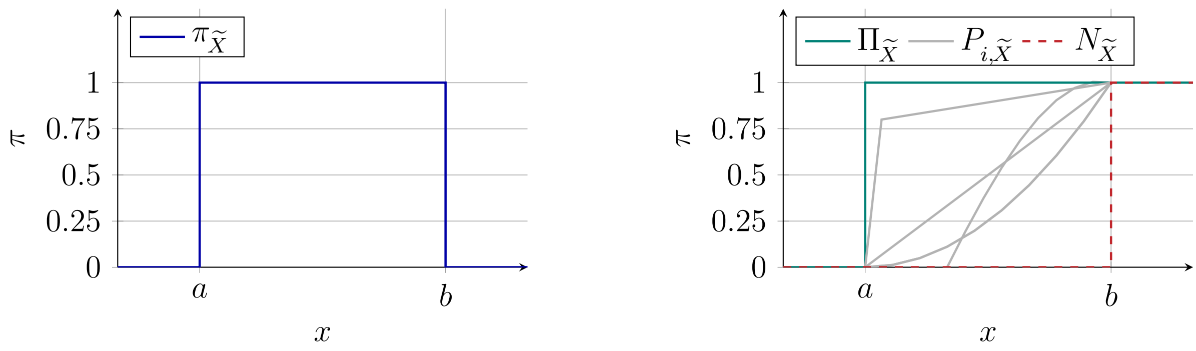
2.2 Combination of information from multiple sources

Whenever imprecise variables describe the outcome of the same event, a conjunction representing the combined body of evidence is of great interest.

Hose [3] showed that for the *special cases* of specificity-ordered, comonotone or quasi-vacuous possibility functions, the conjunction can be calculated using the minimum operator

$$\pi_{\widetilde{X}}^{\text{conj, special}}(x) = \min_{x}\left(\pi_{\widetilde{X}_1}(x), \ldots, \pi_{\widetilde{X}_m}(x)\right). \tag{6}$$

Preserving the conservative validity of the result for the *general case*, however, imposes additional constraints on this calculus. For this, only a conservative bound could be found,



(a) Quasi-vacuous elementary possibility function $\pi_{\widetilde{X}}$

(b) Induced possibility and necessity measure $\Pi_{\widetilde{X}}$ and $N_{\widetilde{X}}$ as well as some consistent cumulative distribution functions $P_{i,\widetilde{X}}$

Figure 1: From the elementary possibility function to the credal set

which has shown to be sufficiently tight in practice [3, p. 45ff]. The universal conjunction of multiple possibilistic variables is then given by

$$\pi_{\widetilde{X}}^{\text{conj, universal}}(x) = \min\left(1, m \cdot \pi_{\widetilde{X}}^{\text{conj, special}}(x)\right), \tag{7}$$

where m is the number of joined variables. Note that this definition for the conjunction matches the one given for the copula of unknown interaction introduced in Eq. (5).

2.3 Monte-Carlo-based description and propagation of uncertainty

For the numerical evaluation of possibilistic calculus a Monte-Carlo-based approach can be used. The main advantages are the ability to describe any shape of distribution in a fully configurable and parallelizable manner, as well as the ability to handle non-linear problems. The core operators of possibilistic calculus, min and sup, which can be hard to put into effect in a continuous domain, can easily be implemented in this way. A possible disadvantage may be the loss of conservativity for insufficiently densely sampled areas, which can be mitigated satisfactorily by the use of appropriate sampling strategies. A distribution is represented by a set of samples called μ-tuples [3]. Every tuple consists of a value x and an assigned membership μ (or π). Extension then becomes as simple as adequately sampling the value range of the input function and evaluating every sample according to the given instruction, while preserving the membership of the sample. The quality of the Monte-Carlo approximation relies on the number of samples but also heavily upon the chosen sampling strategy. A detailed description of sampling strategies, possibilistic propagation using a sample-based approach as well as an optimized method can be found in [7].

2.4 Data sources

Information in the form of possibilistic distributions can be obtained from a variety of sources. On the one hand, precisely known probabilistic distributions can be transformed into possibilistic distributions using the P-Π-transform [3, p. 57f].

On the other hand, in the presence of only limited data, the reliable membership transform [3, p. 139ff] provides a way to obtain a possibilistic confidence distribution from a given small collection of data, i.e. in a setting where a probabilistic distribution cannot be inferred precisely. A highlight of this method is the fact that it provides a configurable level of confidence, even for only a few data points. In a subsequent step, the obtained confidence distribution of the parameters can be propagated to a prediction distribution, returning a conservative estimate of the underlying precise but unknown variable. An example of this workflow is presented in [3, p. 129f].

On the far end of the *spectrum of available priors* [8] lies the case of complete epistemic uncertainty, i.e. the case of an unknown distribution within certain limits. In this case, uncertainty can be possibilistically quantified using a quasi-vacuous distribution [3], as already mentioned in the introductory sections. Information of this type is extremely common in classical engineering and often available in the form of established conservative heuristics of experts, educated guesses, a manufacturer's guarantee, or simply in the form of hard bounds given by the problem domain, such as the room size in a robot localization problem.

Finally, qualitative information, such as plausibility orderings of experts, can also be encoded possibilistically, but are not subject of the investigations in this publication.

2.5 Interpretation of possibilistic distributions

Given an arbitrary type of possibilistic distribution π, one can deduce a Neyman-Pearson confidence interval for the underlying uncertain variable $\widetilde{X}$ through evaluation of its superlevel sets or α-cuts. For the confidence level $1-\alpha$ it is simply given by the superlevel set of α, as shown in [3] and visualized in Figure 2.

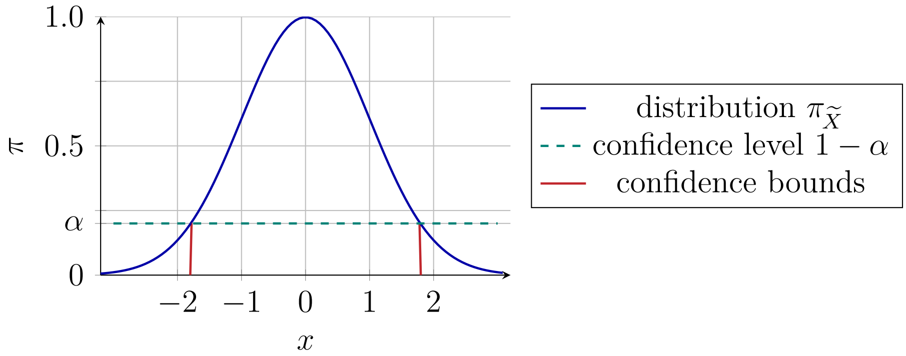


Figure 2: Neyman-Pearson confidence interval [-1.78, 1.78] for confidence level $1-\alpha = 80\%$

3 EXEMPLARY APPLICATION: ROBOT LOCALIZATION

A robot in a test field is equipped with a low-price ultra-wideband setup that can measure the distance to fixed ceiling-mounted beacons with a promised accuracy of 10 to 50 cm. A similar setup is described in [2]. Now suppose the location of these beacons is hard to measure with any degree of accuracy. Only the distance measurements from a handful of known robot positions are available. The task is to locate the robot in future unknown positions based on these initial measurements.

This can be achieved by first inferring the unknown beacon locations B_i from known robot positions R_j and a first batch of uncertain distance measurements $\boldsymbol{q}_0$. Subsequently, after having moved the robot to unknown locations, the robot position Y can be quantified using only its uncertain distance measurements $\boldsymbol{q}$ and the previously inferred beacon locations. A schematic of this process is shown in Figure 3.

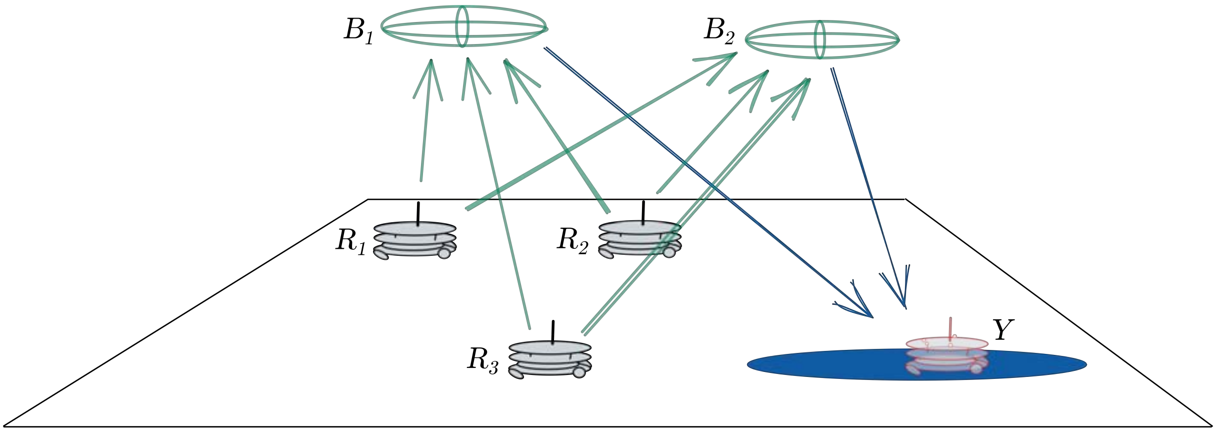


Figure 3: Qualitative setup of the robot localization task. From known robot positions R_j via inferred beacon locations B_i to the unknown robot position Y

Step 1a: Reliable identification of distances

In a first step, the distance measurements have to be converted into possibilistic distributions. It has been verified that the measurement setup produces a normally distributed measurement error. Using the reliable membership transform, parameter distributions and prediction distributions for the distance measurements are calculated, as shown exemplarily for two robot-beacon pairs in Figure 4.

Step 1b: Distance identification with precisely known error distribution

In workflow variant (a), only the normally distributed characteristics of the measurement error are known. Its standard deviation is unknown and has to be reliably inferred from a limited number of measurements. Given more experience or specification on the behavior of the measurement setup, it is certainly valid to also assume a workflow with known standard deviation. In this case, after the intermediate step of parameter identification, the two-dimensional parameter distribution (mean and standard deviation) can be reduced to a one-dimensional distribution of just the mean, which subsequently narrows the distance distribution.

Step 2: Inference of beacon locations

Having obtained possibilistic descriptions of the distance measurements, the unknown beacon locations can be inferred by

$$\pi_{B_i}(b) = \mathcal{J}^{\mathrm{UI}}\left(\pi_{d_{ij}}(||b - R_j||)\right) \quad \forall b \in \mathbb{R}, \quad i = 1, \ldots, m, \quad j = 1, \ldots, n \tag{8}$$

using the copula for unknown interaction $\mathcal{J}^{\mathrm{UI}}$ defined in Eq. (5) and where $R_j \in \mathbb{R}$ is the known robot position j, and $\pi_{d_{ij}}$ is the membership function of the identified distance between beacon i and known robot position j.

Step 3: Update of beacon positions based on prior knowledge

Having inferred possible beacon locations, the beacon positions can be updated by combining the knowledge from the inferred beacon locations with the prior knowledge of

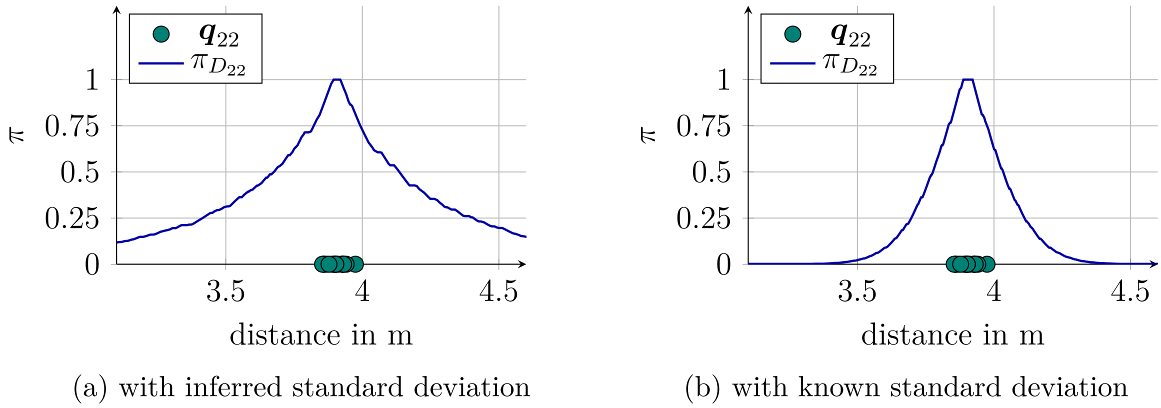


Figure 4: Observed distances from R_2 to B_2 and their possibilistic prediction distributions according to the reliable membership transform. Notice the substantial difference in the width of the two distributions.

the room geometry. The room is represented as the quasi-vacuous distribution

$$\pi_{\text{room}}(b) = \begin{cases} 1 & \text{if } b \in \text{room} \\ 0 & \text{otherwise.} \end{cases} \tag{9}$$

Another contribution of prior knowledge is the fact that the beacons are mounted on the ceiling and the staff can comfortably walk below them, restricting the beacons' position to a height of above 2.20 meters. Furthermore, the ceiling height is known to be 2.70 meters. Mathematically, this can be expressed as

$$\pi_{\text{height}}(b) = \begin{cases} 1 & \text{if } 2.20\,\text{m} \leq b_z \leq 2.70\,\text{m} \\ 0 & \text{otherwise.} \end{cases} \tag{10}$$

Conjunction of the inferred beacon locations, the room geometry and the height information according to Eq. (6) yields the updated beacon locations

$$\pi_{B_i}(b) = \min\left(\pi_{B_i}(b), \pi_{\text{room}}(b), \pi_{\text{height}}(b)\right). \tag{11}$$

Step 4: Calculation of unknown robot positions

Finally, the robot is moved to an unknown position and the distance measurements are taken again. With this new set of measurements $\{q_1, \ldots, q_n\}$ and the inferred and updated beacon positions, the possibility distribution of the unknown robot position Y can be calculated by once again first identifying a description for the distance $\pi_{D_i}(d)$ between the robot and the beacon through the reliable membership transform and then combining the knowledge from each beacon to the robot position by

$$\pi_Y(p) = \min_{i=0,\ldots,3} \sup_{p,b} \mathcal{J}^{\text{UI}}\left(\pi_{D_i}(||p - b||), \pi_{B_i}(b)\right), \tag{12}$$

where $p \in \mathbb{R}^2$ are all points in the ground plane, $b \in \mathbb{R}^3$ are all possible beacon locations, π_{D_i} is the identified prediction distribution of the distance between robot and beacon i and $\mathcal{J}^{\text{UI}}$ is once again the copula for unknown interaction defined in Eq. (5).

Scenarios of different information sources

In order to highlight the modularity of the approach and investigate the impact of the different information sources, three scenarios are considered in the following. In scenario **A**, only the distance measurements between robot and the two beacons, as well as the implicit knowledge of the robot being able to only move in the plane, are available. Scenario **B** adds the knowledge of the room geometry, as well as an approximate range of the beacon heights, as formulated in Eqs. (9) and (10). Finally, in scenario **C**, the sensor standard deviation no longer has to be inferred from limited measurements but is set to a fixed value of 0.02 meters, which is the result of extensive tests with the actual sensor setup in the given lab environment. Comparable values are reported in similar localization setups in the literature [9, 10].

Results

Figure 5 shows the results for the inferred robot position for each scenario. It is evident that the results in scenario **A** do not allow for any meaningful support in the localization of the robot. When making use of the prior knowledge of the room geometry in scenario **B**, the uncertainty of the robot position is reduced slightly, but remains too large to be of any use. Finally, when knowledge about the standard deviation of the distance measurements is added to the body of prior knowledge in scenario **C**, the uncertainty of the robot position is reduced significantly. However, the area for confidence level 95 % still has a size of almost six square meters.

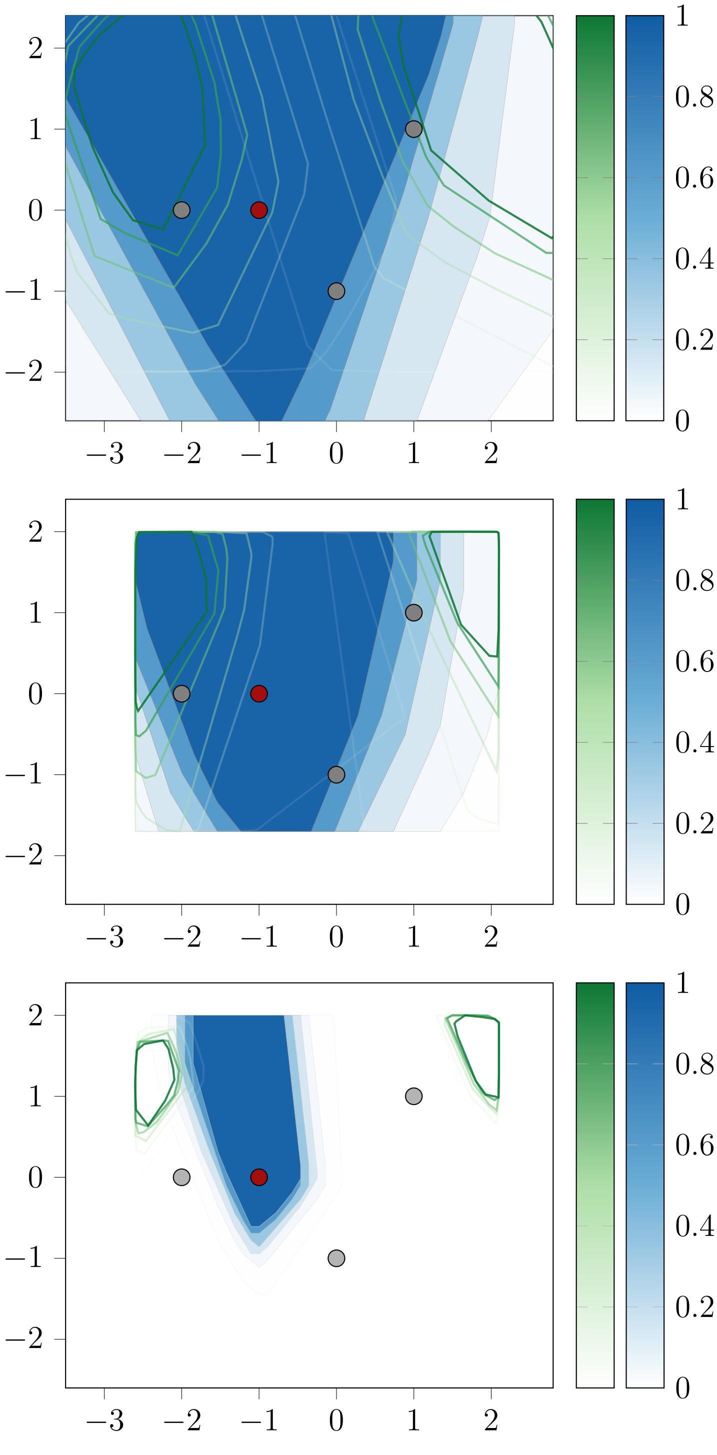
4 CONCLUSIONS

In this contribution, a workflow highlighting the use of possibility theory for uncertainty quantification is presented by means of a localization problem for robotics in a real lab setting. It serves as a contribution towards the application of the framework of possibility theory as a tool for uncertainty quantification in engineering. A highlight of the description of uncertainties in the form of possibility distributions is the conceptually simple integration of prior knowledge, such as room dimensions and sensor characteristics, easily extending and thus enriching an existing workflow.

For this, an efficient propagation strategy in the form of a sample-based method is employed successfully.

A limitation worth discussing are the seemingly wide output distributions with worst-case bounds of at least 1-2 meters when using a sensor setup with advertised accuracies of below 50 centimeters. The possibilistic approach aims at modeling knowledge and its associated uncertainty as faithfully as possible. As such, possibilistic calculus represents a conservative approach to uncertainty propagation, resulting in guarantees for the output distributions given, while showing great flexibility for the type and quantity of information. As a trade-off, a ”broadening” of possibility distributions must often be accepted in order to keep their advantageous properties along the way.

The size and shape of the output distributions can significantly be modified by the number and relative position of nearby UWB beacons. An obstacle to be overcome in future work is the need for less expanding mathematical expressions for the combination of distributions. The current formulation can lead to the paradoxical situation where adding consonant information of unknown interaction to the body of knowledge can lead to a



(a) Scenario **A** with no prior knowledge. The area of the 95 % confidence interval for the unknown robot position mostly exceeds the room size by a large margin. The conservative nature of possibility theory diffused the results to a point of unusability.

(b) Scenario **B** with added knowledge of the room geometry. A reduction in the uncertainty of the robot position can be observed.

(c) Scenario **C** with added knowledge of standard deviation of distance measurements.

Figure 5: Inferred beacon and robot positions for different information scenarios. The inferred robot location is marked in a blue gradient. Beacon positions are projected from 3D space onto the ground plane and marked with a green gradient. All distributions are discretized to 7 levels for improved legibility. Known robot positions are marked in gray for reference.

more conservative distribution, an effect reminiscent of the problem of false confidence [11], which possibility theory was supposed to solve in the first place.

While the used experimental setting is of minimal complexity, the general idea of first inferring the beacon positions and thereafter the robot position applies to any setting where the beacon location is unknown or only measurable with difficulty or uncertainty. This two-way identification also helps alleviate the impact of systematic biases in the sensor measurements.

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