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CERTIFIED INTERVAL MODEL UPDATING USING SCENARIO OPTIMISATION

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Abstract. *Interval model updating is typically performed when limited amounts of data are available to perform non-deterministic model updating. This is especially the case in practical situations, where gathering data is difficult, time-consuming or expensive. In these situations, a metric to certify the accuracy and reliability of the identified interval bounds against missing data could effectively reduce the data-gathering effort. This paper presents a methodology that performs interval model updating and provides such a reliability certificate. Hereto the approach in this paper utilises the recently presented scenario theory of Crespo et al. [6]. The scenario theory rigorously bounds the probability of unseen data falling outside the identified data enclosing set. This probability bound is obtained without an assumption on the underlying distribution and is completely based on the data set size and, in the context of intervals, the number of interval bounds. An interval model updating consists of two steps: (1) gathering the data on the output side of the computational model and (2) updating the input side of the computational model. In this paper, scenario theory is used in the first step to bound the probability of the intervals fitted around the measurement data. In the second step, for a monotonic model, a convex optimisation algorithm in combination with the vertex analysis yields the exact solution up to the computational error. As a result, the final probability of the updated input interval set is completely determined by the first step. The technique is illustrated for the model updating of interval uncertainty in an academic case study of a four-level building.*

Keywords: Inverse uncertainty quantification, scenario optimisation, interval uncertainty.

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1 INTRODUCTION

Interval model updating is a non-deterministic method that is used to update the uncertainty in models based on data. This approach takes the possible range of values for each measurement into account, rather than assuming that the measured values are exact or follow a predefined probabilistic distribution. Taking into account the uncertainty in the model updating is particularly important in cases where the measurements are subject to significant uncertainty or noise. Intervals are especially powerful in cases where gathering data is difficult, time-consuming or expensive [12]. Therefore, several interval model updating procedures have been introduced in literature [8, 11, 7]. Fang et al. [10] used interval response surface models to save computational cost and to limit the typical overestimation of interval model updating approaches. Also, methodologies using a Kriging predictor model were introduced [13, 17]. Faes et al. [9] described a multivariate interval model updating, that is efficient for high dimensional models under scarce data availability. In other situations where gathering data is easy, the analyst can use a probabilistic technique to describe the uncertainty and for the model updating, such as the popular probabilistic model updating technique, Bayesian model updating [1, 16, 19, 2, 18].

In both situations, interval or probabilistic uncertainty, a metric for the accuracy of the identified model uncertainties against unseen missing data would provide the analyst with a tool to assess the “quality” of the identified uncertainty. It also could effectively reduce the data-gathering effort to a minimum. This paper defines the metric as the reliability of a data enclosing set. Here, reliability is defined as the probability that future unseen data will fall in the data-enclosing interval set. Recently, Crespo et al. [6] presented a scenario optimisation approach to provide a minimal value of this reliability without having to characterise the underlying distribution of the data generation mechanism. Hereto, the metric needs the number of hyperparameters to fit the data enclosing set, the number of samples and the dimension of the data enclosing set. Together with a confidence level, the minimal value of the reliability is calculated. Also, an analyst could calculate the number of samples before the measurement campaign required for fitting the data enclosing set with minimal required reliability.

In this paper, the scenario optimisation approach is implemented in an interval model updating algorithm of a dynamic model of a four-level building. Hereto, scenario optimisation is used in the gathering of virtual measurement data. Together with a convex interval model updating procedure that yields the global minimum. he model updating algorithm provides a total reliability estimation of the updated input interval set.

This paper is organised as follows. Section 2 introduces interval analysis and solution techniques based on optimisation and vertex analysis. Section 3 presents the used scenario theory and the application of the method for interval model updating. Finally, section 4 performs the certified interval model updating to a dynamical model of a four-level building, that consists of mass and springs to show the added value of the reliability estimation and the accuracy of the model updating.

2 Interval uncertainty

An interval scalar $x^I \in \mathbb{IR}^{dx}$ represents an uncertain parameter x that has a fixed but unknown value, bounded by x_{min} and x_{max}. The domain of closed, real-valued intervals is denoted by $\mathbb{IR}$. x^I can be defined in different notations, such as $[x_{min}, x_{max}]$ or $[\underline{x}, \overline{x}]$, which is a set of real numbers that satisfies $\underline{x} \leq x \leq \overline{x}$. The midpoint and width of an interval scalar x^I are respectively given by

$$x_m = \frac{\overline{x} + \underline{x}}{2}, \tag{1}$$

$$x_w = \frac{\overline{x} - \underline{x}}{2}. \tag{2}$$

An interval vector $\boldsymbol{x}^I \in \mathbb{IR}^{d_\mathcal{X}}$ with independent interval scalars x_i^I, where $i = 1, \ldots, d_\mathcal{X}$, is defined by the Cartesian product of $d_\mathcal{X}$ interval scalars as $\boldsymbol{x}^I = x_1^I \times x_2^I \times \ldots \times x_{d_\mathcal{X}}^I$. Alternatively, $\boldsymbol{x}^I$ can be represented in set notation as:

$$\boldsymbol{x}^I = \left\{x_1^I \quad x_2^I \quad \ldots \quad x_{d_\mathcal{X}}^I\right\}^T = \left\{\boldsymbol{x} \in \mathbb{R}^{d_\mathcal{X}} | x_i \in x_i^I\right\}, \tag{3}$$

In interval analysis, an interval vector $\boldsymbol{x}^I$ defines the hyper-rectangular input-space of a function $\mathcal{M} : \mathbb{R}^{d_\mathcal{X}} \mapsto \mathbb{R}^{d_d}, \boldsymbol{x} \to \boldsymbol{y}$. $\mathcal{M}$ is represented as a numerical model that consists of d_d deterministic functions $m_i : \mathbb{R}^{d_\mathcal{X}} \mapsto \mathbb{R}, \boldsymbol{x} \to y_i$, where $i = 1, \ldots, d_d$. The output of $\mathcal{M}$ when given the input $\boldsymbol{x}^I$ is represented as a solution set $\boldsymbol{y}^S \in \mathbb{R}^{d_d}$ representing the joint bounds on the model responses $\boldsymbol{y}$ of interest. This set is explicitly given as:

$$\boldsymbol{y}^S = \left\{\boldsymbol{y} | \boldsymbol{y} = \mathcal{M}(\boldsymbol{x}), \boldsymbol{x} \in \boldsymbol{x}^I\right\}. \tag{4}$$

Since finding the exact set $\boldsymbol{y}^S$ in the general case constitutes an NP-hard problem, $\boldsymbol{y}^S$ is usually approximated by an interval vector $\boldsymbol{y}^I \in \mathbb{IR}^{d_d}$: $\boldsymbol{y}^I = \left\{y_1^I \quad y_2^I \quad \ldots \quad y_{d_d}^I\right\}^T$. The components $y_i^I = \left[\underline{y}_i, \overline{y}_i\right]$ of y^I are determined by means of (anti-)optimisation, i.e.,

$$\underline{y}_i = \min_{\boldsymbol{x} \in \boldsymbol{x}^I} \left(m_i(\boldsymbol{x})\right), \tag{5}$$

$$\overline{y}_i = \max_{\boldsymbol{x} \in \boldsymbol{x}^I} \left(m_i(\boldsymbol{x})\right). \tag{6}$$

With this optimisation the interval of each component y_i^I of $\boldsymbol{y}^I$ is found independently, resulting in an approximation of the solution set $\boldsymbol{y}^S$ as a hyper-rectangle. By following this approach, a total of $2d_d$ optimisation problems have to be solved, each potentially requiring numerous model evaluations of $\mathcal{M}$ [14, 8].

Alternatively for a model $\mathcal{M}$ that is monotonic in $\boldsymbol{x}^I$ the interval of each component y_i^I of $\boldsymbol{y}^I$ can also be found by vertex analysis. In this approach, each component is found independently by minimising/maximising the model responses of the set of vertex points $\mathcal{V}$ of $\boldsymbol{x}^I$. This set contains all possible input parameter combinations located at vertex points of the interval input vector $\boldsymbol{x}^I$, yielding $2^{d_\mathcal{X}}$ combinations. Which becomes computationally demanding for high $d_\mathcal{X}$. For high $d_\mathcal{X}$ and linear models the authors recently introduced the Multilevel Monte Carlo method for interval analysis [3].

3 Certified interval model updating

3.1 Reliability of data enclosing sets

In this paper, the reliability of a data-enclosing interval set is defined as the probability that future unseen data will fall in the data-enclosing interval set. The recently presented scenario theory [6] enables bounding this reliability without having to characterise the underlying distribution of the data generation mechanism.

The cost function of a data-driven constrained optimisation is defined as $\mathcal{J} : \Theta \rightarrow \mathbb{R}$ with design variable $\theta \in \Theta \subset \mathbb{R}^{n_\theta}$. Where n_θ is the number of design variables. Also, denote the set of design points satisfying the design requirements for scenario $\delta \in \Delta$ as h_δ. Then consider the constrained, data-driven scenario program

$$\theta^*(\mathbf{D}) = \underset{\theta \in \Theta}{\operatorname{argmin}} \left\{ \mathcal{J}(\theta) : \theta \in \bigcap_{i=1}^{n} h_{\delta^{(i)}} \right\}, \tag{7}$$

where the data $\mathbf{D}$ is sampled from a stationary data generating mechanism. The unknown probability measure governing the underlying data generation mechanism is defined as P. The reliability of θ^* is $1 - V(\theta^*)$, with $V(\theta^*)$ being the violation defined as

$$V(\theta^*) = P\left[\delta \in \Delta | \theta^* \notin h_\delta\right]. \tag{8}$$

The lower the violation, the higher the reliability of θ^*. As the data set $\mathbf{D}$ is chosen randomly out of infinitely many possible data sets of size n, the θ^* and thus also $V(\theta^*)$ is random. This randomness can be quantified by using

$$P^n\left[V(\theta^*) \leq \epsilon\right] \geq 1 - \beta, \tag{9}$$

this equation defines that the probability $P^n = P \times \cdots \times P$ of the violation of θ^* being less or equal than $\epsilon \in [0, 1]$ is greater than $1 - \beta$. Herein is $\beta \in [0, 1]$ the confidence and ϵ called the reliability. Note that, θ^* is a random element that depends on n randomly chosen samples from P. Therefore, the violation probability $V(\theta^*)$ can be greater than ϵ for some random observations but not for others, and β refers to the probability P^n of observing one of those bad sets of n samples. As a result, the confidence β is thus key to obtain results that are guaranteed independently of P.

As a result, scenario theory allows evaluating (9) without making any assumption about P. When the optimisation program (7) is convex, the hyperparameters are obtained with a convex optimisation from the data set, $V(\theta^*)$ is dominated by a beta distribution [5, 4], and ϵ can be calculated from

$$\binom{k + n_\theta - 1}{k} \sum_{i=0}^{k+n_\theta-1} \binom{n}{i} \epsilon^i (1-\epsilon)^{n-i} \leq \beta, \tag{10}$$

where $k < n - n_\theta$ is the number of data points/outliers removed from the data set $\mathbf{D}$ before θ^* was calculated. Equation (10) allows the analyst to compute a reliability estimate before θ^* is computed. As a result, the reliability can be calculated before data is drawn from the data generation mechanism [6].

3.2 Reliability for interval fitting

Fitting an interval around an dataset $\mathbf{D} \in \mathbb{R}^{d_\delta}$ containing n_δ samples δ in d_δ dimensions is defined as

$$\mathbf{D}^{\boldsymbol{I}} = \begin{bmatrix} \underline{\delta_1} = min(\delta_1) & \overline{\delta_1} = max(\delta_1) \\ \vdots & \vdots \\ \underline{\delta_{d_\delta}} = min(\delta_{d_\delta}) & \overline{\delta_{d_\delta}} = max(\delta_{d_\delta}) \end{bmatrix}. \tag{11}$$

For each dimension d_δ of δ 2 interval bounds are calculated. As a result, the total number of design variables is $n_\theta = 2 * d_\delta$.

A reliability certificate calculated with equation 10 requires a random data sampling strategy and a convex optimisation procedure to find all θ or for this case D^I. Fitting intervals satisfies this requirement as finding the interval set of a randomly sampled data set yields the exact interval bounds for that data set.

3.3 interval model updating

Interval model updating is the process of updating the input interval vector $\boldsymbol{x}^I$ of the numerical model $\mathcal{M}$ based on the output $\boldsymbol{y}$ of the numerical model and measurement data $\boldsymbol{y}_m$. In this paper, the interval model updating minimises the distance between the fitted output interval vector $\boldsymbol{y}^I$ of the numerical model and the fitted interval vector that bounds the measurement data $\boldsymbol{y_m}^I$. Hereto, the interval vector of the output $\boldsymbol{y}^I$ is calculated with equations 5 and 6 or another interval analysis approach discussed in section 2 [15, 3]. The interval vector of the measurement dataset $\boldsymbol{y_m}^I$ is calculated with equation 11. From these intervals the midpoints ($\boldsymbol{y}_m$, $\boldsymbol{y_m}_m$) and widths ($\boldsymbol{y}_w$, $\boldsymbol{y_m}_w$) are calculated with respective equations 1 and 2. The midpoints and width are used for the calculation of the distances between the measurement interval set $\boldsymbol{y_m}^I$ and the output interval set $\boldsymbol{y}^I$. minimisation is the sum of the The error function of the optimisation is defined as

$$error = \left(\frac{||\boldsymbol{y}_m - \boldsymbol{y_m}_m||_2}{2||\boldsymbol{y}_{mi} - \boldsymbol{y_m}_m||_2} + \frac{||\boldsymbol{y}_w - \boldsymbol{y_m}_w||_2}{2||\boldsymbol{y}_{wi} - \boldsymbol{y_m}_w||_2} \right), \tag{12}$$

where $||\cdot||_2$ is the 2 norm and $\boldsymbol{y}_{mi}$, $\boldsymbol{y}_{wi}$ the midpoint and width of an initial guess before the optimisation is started. The error function is split into two parts, one on the midpoint and one on the width. Both parts are starting with an error of 0.5 so that the goal function starts at 1. The error function yields zero when ($\boldsymbol{y}_m$, $\boldsymbol{y_m}_m$) are equal and ($\boldsymbol{y}_w$, $\boldsymbol{y_m}_w$) are equal. The optimisation is defined as

$$\boldsymbol{x}^{I,*} = \underset{x^I \in \mathbb{IR}^{d_{\mathcal{X}}}}{\text{argmin}}(error). \tag{13}$$

When the minimisation is convex a gradient-based optimisation procedure yields the global minimum up to a numerical error. One of the requirements for a convex optimisation is using a numerical model that has a monotonic response to the input intervals. A global minimum is not guaranteed for non-convex minimisation.

3.4 certified interval model updating

The certificate of an interval model updating procedure is defined in this paper as, the minimal reliability ϵ_{tot} for the resulting interval set $\boldsymbol{x}^{I,*}$ of the interval model updating procedure against unseen data. To obtain this reliability, the total reliability of the interval model updating procedure needs to be calculated. Hereto, the interval model updating procedure is split into two parts (1) gathering the measurement data and fitting the interval set around it $\boldsymbol{y_m}^I$, (2) performing the optimisation and finding the input interval set $\boldsymbol{x}^{I,*}$. For the first step, the measurement interval is fitted with 11 and the reliability ϵ_m calculated with 10. For the second step, the solution of convex optimisation yields an exact solution. As a result, the reliability of a convex optimisation is $\epsilon_o = 100\%$. When this reliability is assumed to be independent and have the same confidence level, the total reliability equals the product of both individual reliability levels

$$\epsilon_{tot} = \epsilon_m * \epsilon_o. \tag{14}$$

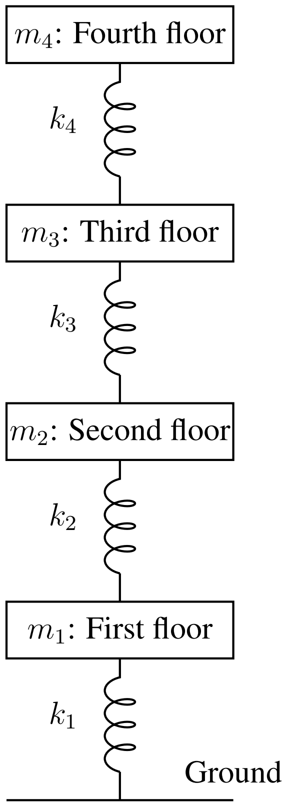

Figure 1: Dynamic model of a four-level building with masses $\{m_1, m_2, m_3, m_4\}$ and stiffness values $\{k_1, k_2, k_3, k_4\}$.

4 Updating a dynamic model of a four-level building

The model used in the case study is a four-level building that is visualised in 1. In the model, the mass of the four levels is equal to $m = 4000$ kg $= m_1, m_2, m_3, m_4$. The stiffness values k_1, k_2, k_3, k_4 are different for each connection between the floors and defined as interval uncertainty. The interval vector $\boldsymbol{k}^I$ is defined as

$$k_1^I = [4030 \ \ 4070] \text{ N/m},$$

$$k_2^I = [3940 \ \ 4100] \text{ N/m},$$

$$k_3^I = [3980 \ \ 4080] \text{ N/m},$$

$$k_4^I = [4060 \ \ 4120] \text{ N/m}.$$

and from which a data set is randomly sampled in subsection 4.1. During the optimisation, the mode shapes are tracked with the Model Assurance Criterion (MAC). This is to ensure that the compared eigenvalues are linked with the same mode shape and when a mode switch is detected the corresponding eigenvalues are switched as well.

4.1 Numerically generated data set

Data is generated with a numerical dynamic model as visualised in figure 1. Before the input interval set is sampled, the minimal number of samples is calculated with equation 10. Hereto defining the following parameters is a requirement, the minimal reliability ϵ, the confidence of the reliability β, the number of outliers that will be removed k and the number of design variables n_θ. The goal of this interval model updating is to obtain a 95% reliability on the final updated input vector $\boldsymbol{x}^{I,*}$. To obtain this, the reliability of the numerically generated data must be $\epsilon = 0.95$ with high confidence $\beta = 0.999$. No outliers will be removed as the numerical simulation is a simple analytical model $k = 0$. From the input samples, output samples of the first 4 eigenvalues ϕ_i with $i \in 1, 2, 3, 4$ are generated with the numerical model. Around these output samples, an interval set $\boldsymbol{\phi}^I$ which components $\underline{\phi_i}, \overline{\phi_i}$ are fitted with

$$\overline{\phi_i} = \max(\phi_i), \tag{15}$$
$$\underline{\phi_i} = \min(\phi_i). \tag{16}$$

The output interval set defines the smallest independent hyper-rectangle around the output samples. The interval fitting approach of equation 15 and 16 needs in total $n_\theta = 8$ design parameters to be calculated as defined in 3.2. With the reliability, confidence, number of outliers and number of design variables known. The minimum number of samples is calculated with equation 10 to get at least the required reliability and confidence. This is 386 random samples. Note that this random sampling on the input side is only for generating/simulating random responses on the output side that an analyst could see in experiments and is also a requirement for the reliability calculation of the data enclosing set. From these samples the obtained output vector is

$$\begin{bmatrix} \underline{\phi_1} = 0.0594 \text{ Hz} & \overline{\phi_1} = 0.0609 \text{ Hz} \\ \underline{\phi_2} = 0.1776 \text{ Hz} & \overline{\phi_2} = 0.1804 \text{ Hz} \\ \underline{\phi_3} = 0.2702 \text{ Hz} & \overline{\phi_3} = 0.2771 \text{ Hz} \\ \underline{\phi_4} = 0.3255 \text{ Hz} & \overline{\phi_4} = 0.3329 \text{ Hz} \end{bmatrix}. \tag{17}$$

4.2 Interval Model Updating

The interval model updating uses an optimisation program to minimise the error between the simulated generated data set and the propagated data set based on the to-be-updated input intervals. The minimised error function is equation 12. Each iteration of the model updating uses vertex analysis to propagate the input interval vector $\boldsymbol{x}^I$ to the output interval vector $\boldsymbol{y}^I$ through the numerical model. This is possible as the eigenvalue analysis of the numerical model is monotonic.

The input interval vector is updated based on the midpoints and widths of its containing interval scalars. The possible ranges where the optimisation is allowed to vary the interval widths and midpoints are $[90 \;\; 110]$ for the widths and $[3900 \;\; 4100]$ for the midpoints. These ranges are then scaled to be between $[0 \;\; 1]$.

An advantage of the vertex analysis is that the exact interval bounds are obtained and that a convex optimisation is obtained. As a result, the interval model updating can use a gradient descent algorithm of the type SQP, with a stopping criterion of maximum 5000 function evaluations and a step tolerance of $1e^{-18}$. The convex optimisation yields a resulting interval input vector $\boldsymbol{x}^{I,*}$ up to the numerical precision and with 100% reliability with 100% confidence. Together with the reliability of the interval set fitted around the numerically generated dataset $\boldsymbol{y_m}^I$ of 95% with 99% confidence the total reliability is calculated with equation 14 and is 95% with 99% confidence.

4.3 Results

The optimisation algorithm of the interval model updating stops after 1446 model evaluations, with a remaining error of $3.78e^{-8}$. Figure 2 shows the interval bounds on the output side,

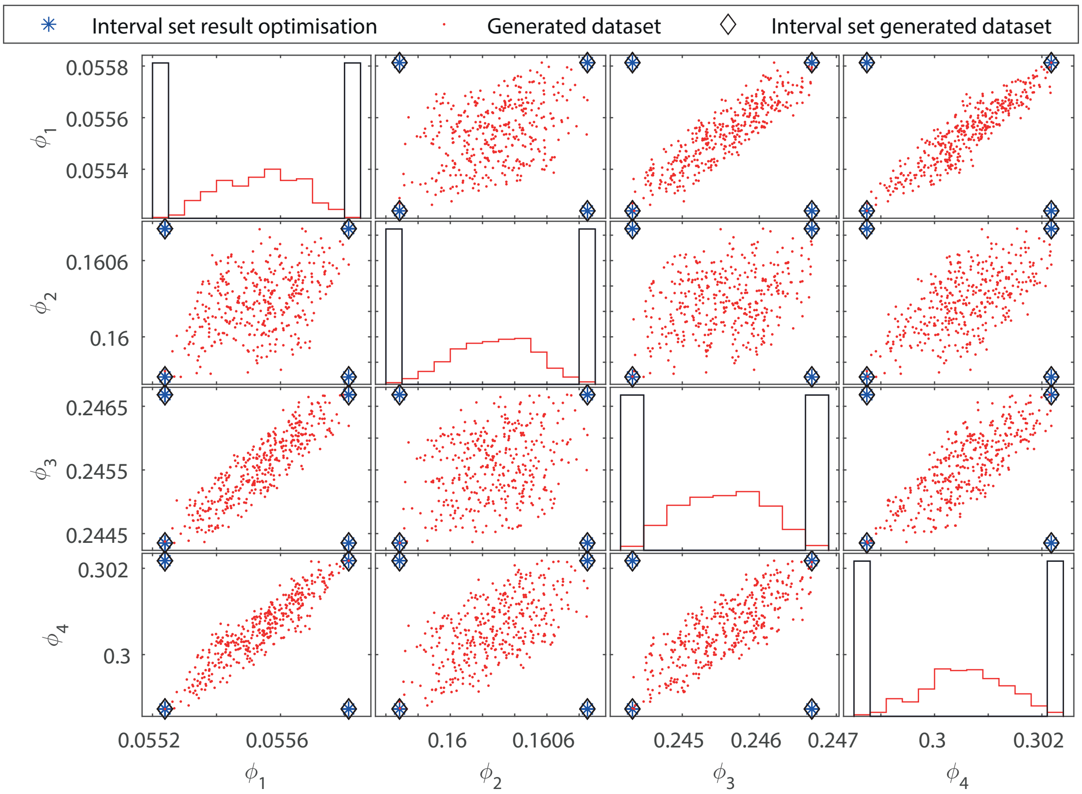

Figure 2: The numerically generated data set in red "·" with the fitted interval set in black "◊" and the set of the propagated solution in blue big dots "∗".

the propagated numerically generated data set $\boldsymbol{y}_m$ with its interval set bounds $\boldsymbol{y}_m^I$ and the interval set of the propagated solution of the optimisation algorithm. From this figure and the low remaining error, it is clear that in the output space, the optimisation found an optimum.

The obtained interval set on the input side $\boldsymbol{x}^{I,*}$ is summarized in table 1 together with the initially defined interval set $\boldsymbol{k}^I$ to generate the numerical data. This shows that the maximal error on the obtained bounds is 0.425%, with the certificate that the reliability of the obtained interval bounds is at least 95% with 99% confidence against missing unseen data.

Table 1: Comparison of the obtained stiffnesses and the stiffnesses used for data generation all units are N/m

Stiffness	defined input interval	bounds on input samples	model updating result
k_1^I	$[4030\ \ 4070]$	$[4030\ \ 4070]$	$[4029\ \ 4074]$
k_2^I	$[3940\ \ 4100]$	$[3941\ \ 4099]$	$[3948\ \ 4083]$
k_3^I	$[3980\ \ 4080]$	$[3980\ \ 4080]$	$[3982\ \ 4069]$
k_4^I	$[4060\ \ 4120]$	$[4060\ \ 4120]$	$[4071\ \ 4113]$

5 CONCLUSIONS

The paper presents a certified interval model updating procedure, that performs an interval model updating where a certificate against unseen missing data is provided. Hereto the approach in this paper utilises scenario theory to rigorously bound the probability of unseen data falling outside an identified data enclosing set. The reliability approach provides the analyst with additional answers, for instance, whether enough data is generated to obtain certain reliability of the results, and what the reliability is when only very limited data is available. In this paper, the approach is applied to a modal analysis of a 4-level building to showcase the reliability certificate and the interval model updating of a numerical model. During this case, the mode shapes were tracked with a Modal Assurance Criterion, to make sure no mode switching is occurring. For 4 input intervals a reliability of at least 95% with 99% confidence is obtained with 386 input samples on the identified intervals. This reliability is obtained from the 95% reliability of the fitting interval set around the 386 input samples, and the 100% reliability of the convex optimisation in combination with the vertex analysis during the model updating procedure which yields results up to a numerical error. The results of the case show a small error of max 0.425% which is originating from the accuracy of the optimisation procedure. This shows the practical added value of the reliability certificate and the accuracy of the obtained results. In future work, the methodology could be extended towards other interval model updating strategies and applied to a case with measurement data to validate the approach.

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