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MODEL DISTANCE BASED MULTIVARIATE GLOBAL-LOCAL RESPONSE SENSITIVITY ANALYSIS FOR UNCERTAIN SYSTEMS

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Abstract

The design and analysis of engineering systems involve complex numerical simulations. Generally, the dynamics of interest in these engineering systems involve uncertainties associated with them. The uncertainty in the response quantities of interest can be attributed due to the various sources of input parameter uncertainties. There is a need to quantify these uncertainties in the process of engineering decision-making. Using the notion of global response sensitivity analysis (GRSA), the uncertainties associated with the response quantity of interest can be quantified in terms of the input uncertainties. For scalar response quantity of interest, the traditional approaches for GRSA, such as Sobol's analysis and moment-independent sensitivity analysis, can be used. This paper extends the idea of performing sensitivity analysis for systems with multiple responses. The study deals with the computation of global-local response sensitivity indices (GLRSIs) for systems with multiple outputs when the uncertainties in the input variables are modelled probabilistically. The definition of GLRSIs is based on the notions of distance measures between a fiducial response model in which all the uncertainties are present and an altered response model in which certain specified sources of uncertainty are deliberately suppressed. The distance between these two response models is taken as a measure of sensitivity with respect to the variable whose uncertainty has been suppressed. Here, the probabilistic distance measures defined for random vectors are used, which involve the definitions based on joint probability density functions of the random vectors. The study employs different probabilistic distance measures (such as those based on the Hellinger metric and l_2 norm) to achieve this. The significant input variables can be identified by the relative ranking of these distance-based sensitivity measures. The computation of these indices is based on the application of Monte Carlo simulation approaches. Illustrative examples include studies on a nonlinear response function and an uncertainly parametered truss carrying uncertain loads.

Keywords: Multivariate outputs, Moment-independent sensitivity analysis, Probabilistic distance measures, Sobol's analysis.

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1 INTRODUCTION

Mathematical and computational models are commonly utilized to simulate real-world phenomena and predict future outcomes. However, these models are subjected to various types of uncertainty, such as parameter, model, and data uncertainties which can be classified as either aleatory or epistemic uncertainty [1]. Probabilistic theory can be used to model aleatory uncertainty, whereas non-probabilistic theories such as interval, convex, or fuzzy concepts can be used to model epistemic uncertainty [2]. The uncertainties present in the input variables affect the output of the model. To quantify the uncertainties in the output, sensitivity analysis is performed. Sensitivity analysis helps to identify the most important sources of variability and their impact on the model output.

Saltelli [3] defined the sensitivity analysis as the study of how uncertainty in the output of a model can be attributed to different sources of uncertainty in the model input. Many domains, such as reliability analysis, reliability optimization, and reliability design, use sensitivity analysis [4]. Local sensitivity analysis (LSA) and global sensitivity analysis (GSA) are two primary categories of methods used in sensitivity analysis. LSA approaches only reveal information about how inputs affect output at a certain point in the input space [3]. The partial derivatives of the output with respect to the inputs serve as the foundation for these techniques. GSA methods quantify the impact of input variable variations across the entire input space on the output, to determine quantitative measures**.** GSA is an effective tool for identifying significant input variables, evaluating the relationships among input variables, assessing robustness of the model and uncertainty, and improving comprehension of the model and design [5]. By achieving these goals, global sensitivity analysis can enhance the model's dependability, accuracy, and decision-making capability.

Traditional GSA includes non-parametric method [6], elementary effect method [7], variance-based method [8], and entropy-based method [9]. Sobol's method is another widely used approach in GSA. It is a variance-based method that provides a decomposition of the output variance into contributions from individual input parameters and their interactions [10]. Sobol's method has been extensively studied and has been applied to various types of models with scalar output, including static and dynamic models [11]. To reduce the computing costs associated with doing GSA, Kucherenko *et.al.*, [12] suggested using derivative-based global sensitivity indices. These indices were connected to Sobol's sensitivity indices.

Borgonovo *et.al.*,[13] proposed that a model user would benefit more from a metric that takes the output distribution into consideration as compared to certain output moments. Chun *et.al.*, [14] gave the Minkowski distance metric based on the cumulative distribution functions of the model's output. Borgonovo [15] identified a new measure called the moment-independent uncertainty indicator, which considers the entire input/output distribution as well as correlations between variables. Liu and Homma [16] developed a new importance measure for computing a moment-independent uncertainty importance measure based on a double-loop Monte Carlo simulation (MCS), which is a metric for determining the contribution of uncertain inputs to the overall uncertainty of the model output. Greegar and Manohar [17] proposed model distance-based sensitivity analysis by employing statistical measures such as Hellinger distance, Kullback-Leibler divergence, and l_2 norm. The authors also showed that l_2 norm based indices are related to Sobol's indices. Nandi and Singh [18] used statistical measures, such as Kantorovich-Rubinstein metric, Hellinger distance, total variation distance, Kolmogorov, Bhattacharya, and Cramer-von Mises for single variate response.

In the context of models with multivariate output, Campbell [19] observed that conducting sensitivity analysis for each output separately is straightforward. However, this approach duplicates conventional GSA methods and disregards the interdependence among outputs.

Therefore, performing a comprehensive sensitivity analysis may be challenging when analyzing each output separately. Zhang *et.al*., [20] used variance decomposition method for performing GSA of multivariate output. There are many methods available for GSA of multivariate output, such as multivariate analysis of variance (MANOVA) [21], correlation-based sensitivity analysis, [22] and probability distribution function-based approach [23]. As an extension of Sobol's indices, Gamboa *et.al*., [24] identified a set of multivariate global sensitivity indices, which is based on covariance decomposition method. Lamboni *et.al*., [25] proposed a global sensitivity index for multivariate outputs using principal components analysis. As the covariance decomposition method and principal component analysis just consider variance of the output, they will fail when the output uncertainties are not completely represented by the variance. Lijie et al. [26] introduced a novel index for performing the multivariate global sensitivity analysis, which is based on joint probability density functions (jpdfs). However, they encountered difficulties in determining the jpdf of dynamic output. Li et al. [27] proposed a new global sensitivity measure for multivariate output using probability integral transformation. This method involves transforming any continuous distribution of a random variable to a standard uniform distribution by utilizing the cumulative distribution function.

The present study was motivated by the following observations:

1. Utilizing statistical distance measures for a single variate response provides an opportunity to extend this concept to multivariate sensitivity analysis using moment-independent techniques.
2. To the best of the author's knowledge, there has been no previous attempt to perform multivariate sensitivity analysis by utilizing a model distance-based sensitivity measure to consider different probability distribution types of correlated random variables as input parameters. This observation has sparked interest in analyzing structural problems with uncertain parameters.

The objective of this paper is to propose computational methods for assessing global-local sensitivity measures using model distance-based sensitivity analysis for multivariate response. To demonstrate the proposed concepts, this study considers a nonlinear response function and a truss carrying uncertain loads with uncertain parameters.

2 MODEL DISTANCE-BASED SENSITIVITY ANALYSIS FOR SINGLE VARIATE RESPONSE

Consider a deterministic function $Y = f(\boldsymbol{X})$, where the function f could be the result of a simulation of a structural model or the result of a mathematical model and Y is the output quantity of interest. $\boldsymbol{X} \in \mathbb{R}^n$ is a vector of n random inputs (i.e., $\boldsymbol{X} = [X_1, X_2, \dots, X_n]^t$). Here, the superscript t represents transposition of a vector. The jpdf of $\boldsymbol{X}$ is represented by $p_{\boldsymbol{X}}(\boldsymbol{x})$, and the marginal probability density function (pdf) of $X_i; i = 1,2, \dots, n$ can be written as $p_{X_i}(x_i) = \int \cdots \int p_{\boldsymbol{X}}(\boldsymbol{x}) \prod_{k=1,k\neq i}^{n} dx_k$. The output response, Y can be considered as the reference model in which all the input parameters of $\boldsymbol{X}$ are treated as uncertain. The pdf of Y is denoted by $p_Y(y)$.

Consider a hypothetical model, in which all the input parameters are treated as uncertain except the i^{th} input parameter, X_i which has been deliberately fixed at a deterministic value denoted by α_i. Let this model be called altered model-1 and is shown in Figure 1 (a). The output of this altered model is denoted by $\tilde{Y}_i$ and the corresponding pdf is given by $p_{\tilde{Y}_i}(y)$. The difference between the reference model, Y and altered model, $\tilde{Y}_i$ represents the importance of the eliminated uncertainty in X_i and its interactions with other input variables. Let this difference be denoted by $d_i = \text{dist}(Y, \tilde{Y}_i)$ and is a measure of total effect of the input variable, X_i. Larger value of d_i implies higher importance of the uncertainty in X_i towards the output response.

In a similar manner, an altered model-2 can be defined where except for the i^{th} element, X_i, all other variables are made deterministic by fixing them to constant values, $\alpha_j; j = 1,2,\cdots,n; j \neq i$ as shown in Figure 1 (b). Let the outcome of this altered model is denoted by $\tilde{Y}_{\sim i}$ and the corresponding pdf is given by $p_{\tilde{Y}_{\sim i}}(y)$. The difference between the reference model, Y and the altered model, $\tilde{Y}_{\sim i}$ represents the sensitivity of the uncertainty due to X_i alone. This sensitivity measure denoted by $d_{\sim i} = \text{dist}\left(Y, \tilde{Y}_{\sim \text{i}}\right)$ can be considered as the main effect due to variable X_i. Here interaction of X_i with other input parameters are not considered. Higher value of $d_{\sim i}$ represents lower sensitivity effect due to variable X_i.

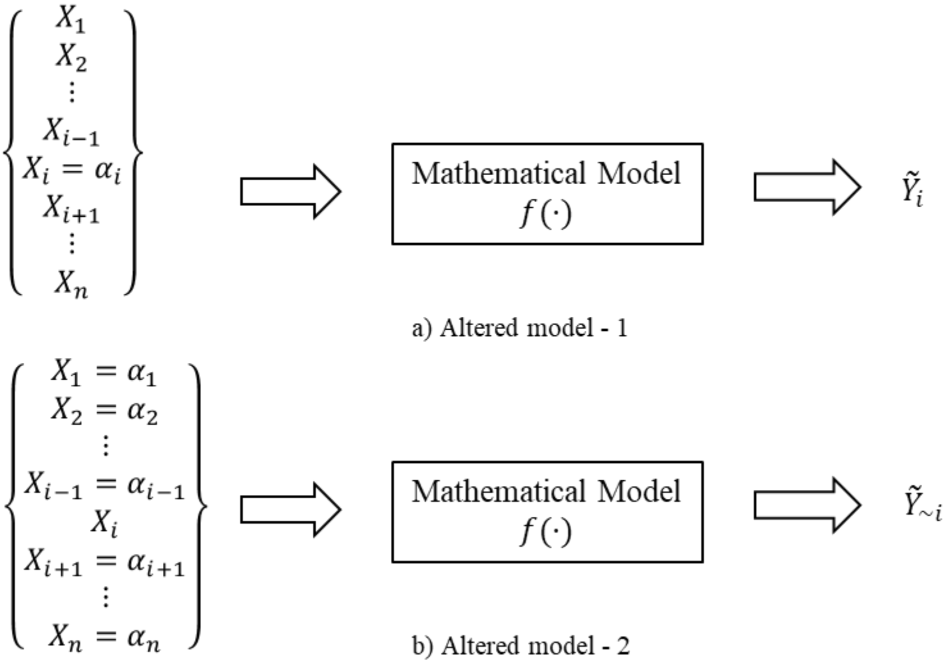

Figure 1: Representation of altered models 1 and 2.

The difference between the reference model and the hypothetical models can be measured by comparing the shift between its probability density/distribution functions using various statistical measures. The different statistical measures are available in literature to measure this disparity [18]. Some of the significant measures are mentioned in Table 1.

Sl. No	Statistical distance measures	Description
1	Hellinger distance	$\sqrt{\frac{1}{2}\int_{-\infty}^{\infty}\left(\sqrt{p_U(u)} - \sqrt{p_V(u)}\right)^2 du}$
2	Kullback-Leibler divergence	$\int_{-\infty}^{\infty} p_U(u)\, ln\left(\frac{p_U(u)}{p_V(u)}\right) du$
3	l_2 norm	$\sqrt{\text{E}[(U-V)^2]}$
4	Total variation distance	$0.5\int_{-\infty}^{\infty} \lvert p_U(u) - p_V(u)\rvert du$
5	Kantorovich-Rubinstein metric	$\int_{-\infty}^{\infty} \lvert p_U(u) - p_V(u)\rvert du$

Table 1: Mathematical expressions of different statistical distance measures used to compare pdfs.

In Table 1, U and V represent the random variables with pdfs, denoted by $p_U(u)$ and $p_V(u)$ respectively. E[·] denotes the mathematical expectation operator.

3 MODEL DISTANCE-BASED SENSITIVITY ANALYSIS FOR MULTIVARIATE RESPONSE

The present study focuses on extending the idea of model distance-based sensitivity analysis for multivariate response using statistical distance measures. Consider a deterministic function, $f: \boldsymbol{X} \rightarrow \boldsymbol{Y}$, given by $\boldsymbol{Y} = f(\boldsymbol{X})$, where the output $\boldsymbol{Y} \in \mathbb{R}^m$ is a vector of m outputs denoted by Y_1, Y_2, ..., Y_m. The notions of altered models 1 and 2 discussed in previous section can be extended here for the multivariate response function in a similar manner. Let $\boldsymbol{Y}$ represents the reference model where uncertainties in all the input variables are retained. Let $p_{\boldsymbol{Y}}(y_1, y_2, \cdots, y_m)$ denotes the jpdf of $\boldsymbol{Y}$. The altered model, defined by fixing the i^{th} input variable to a reference value while keeping all other input variables uncertain, be given by $\widetilde{\boldsymbol{Y}}_i$ and the corresponding jpdf be denoted by $p_{\widetilde{\boldsymbol{Y}}_i}(y_1, y_2, \cdots, y_m)$. The comparison of these two random vectors, denoted by $\text{dist}(\boldsymbol{Y}, \widetilde{\boldsymbol{Y}}_i)$ can be made by measuring the shift between the jpdfs, $p_{\boldsymbol{Y}}(y_1, y_2, \cdots, y_m)$ and $p_{\widetilde{\boldsymbol{Y}}_i}(y_1, y_2, \cdots, y_m)$. This is a measure of total effect with respect to the input variable, X_i. In a similar manner, altered model-2 can be defined by fixing all the input variables except the i^{th} variable. Let this model be denoted by $\widetilde{\boldsymbol{Y}}_{\sim i}$ and its corresponding jpdf by $p_{\widetilde{\boldsymbol{Y}}_{\sim i}}(y_1, y_2, \cdots, y_m)$. $\text{dist}(\boldsymbol{Y}, \widetilde{\boldsymbol{Y}}_{\sim i})$ compares the shift between the random vectors $\boldsymbol{Y}$ and $\widetilde{\boldsymbol{Y}}_{\sim i}$ and is measure of the main effect with respect to the variable, X_i.

The statistical measures listed in Table 1 are applicable in this context for defining the total and main effects by considering the jpdfs of $\boldsymbol{Y}$, $\widetilde{\boldsymbol{Y}}_i$, and $\widetilde{\boldsymbol{Y}}_{\sim i}$. For instance, the Hellinger distance and l_2 norm defined for finding the shift between $\boldsymbol{Y}$, and $\widetilde{\boldsymbol{Y}}_i$, are given by Eqs. (1) and (2).

$$d_H(\boldsymbol{Y}, \widetilde{\boldsymbol{Y}}_i) = \sqrt{\frac{1}{2}\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left(\sqrt{p_{\boldsymbol{Y}}(y_1, y_2, \cdots, y_m)} - \sqrt{p_{\widetilde{\boldsymbol{Y}}_i}(y_1, y_2, \cdots, y_m)}\right)^2 \prod_{i=1}^{m} dy_i} \quad (1)$$

$$l_2(\boldsymbol{Y}, \widetilde{\boldsymbol{Y}}_i) = \sqrt{\text{tr}\left[\text{E}\left[(\boldsymbol{Y} - \widetilde{\boldsymbol{Y}}_i)(\boldsymbol{Y} - \widetilde{\boldsymbol{Y}}_i)^t\right]\right]} \quad (2)$$

Here, tr[·] represents the trace of a matrix. Eqs. (1) and (2) represent the total effects evaluated with respect to the input variable, X_i. In a similar manner, the main effects with respect to the input variable, X_i can also be defined by using $d_H(\boldsymbol{Y}, \widetilde{\boldsymbol{Y}}_{\sim i})$ and $l_2(\boldsymbol{Y}, \widetilde{\boldsymbol{Y}}_{\sim i})$.

3.1 Procedure for estimating the sensitivity indices using brute force method

The process of estimating the sensitivity indices for multivariate output is given below.

1) Simulate N samples of the input random vector $\boldsymbol{X}$ from the jpdf, $p_{\boldsymbol{X}}(x_1, x_2, \cdots, x_n)$ using MCS. Let these samples be denoted by $\left[x_1^{(j)}, x_2^{(j)}, \ldots, x_n^{(j)}\right]^t; j = 1,2, \ldots, N$.
2) Evaluate the response of the reference model, $\boldsymbol{Y} = f(\boldsymbol{X})$ for these N samples of $\boldsymbol{X}$. Let the realizations of the random vector $\boldsymbol{Y}$ be denoted by $\left[y_1^{(j)}, y_2^{(j)}, \ldots, y_m^{(j)}\right]^t; j = 1,2, \cdots, N$. Here $y_k^{(j)}; k = 1,2, \cdots, m$ represents the j^{th} realization of the random variable, Y_k.
3) Evaluate the response of the altered model-1, $\widetilde{\boldsymbol{Y}}_i$. Here the i^{th} input random variable, X_i is kept deterministic by equating to a constant, α_k while keeping other input random

variables of $\boldsymbol{X}$ as uncertain. $\left[\tilde{y}_1^{(j)}, \tilde{y}_2^{(j)}, \dots, \tilde{y}_m^{(j)}\right]^t = f\left(x_1^{(j)}, x_2^{(j)}, \dots, x_{k-1}^{(j)}, \alpha_k, x_{k+1}^{(j)}, \dots, x_n^{(j)}\right); j = 1,2,\dots,N$. Here $\left[\tilde{y}_1^{(j)}, \tilde{y}_2^{(j)}, \dots, \tilde{y}_m^{(j)}\right]^t$, $j = 1,2,\cdots,N$; are the realizations of the random vector $\widetilde{\boldsymbol{Y}}_i = [\tilde{Y}_1, \tilde{Y}_2, \dots, \tilde{Y}_m]^t$.

4) Evaluate the response of the altered model-2, $\widetilde{\boldsymbol{Y}}_{\sim i}$ in which except for the random variable X_i, which is permitted to retain uncertainty, all the other random variables of $\boldsymbol{X}$ are fixed at deterministic values .$\left[\tilde{y}_{\sim 1}^{(j)}, \tilde{y}_{\sim 2}^{(j)}, \dots, \tilde{y}_{\sim m}^{(j)}\right]^t = f\left(\alpha_1, \alpha_2, \dots, \alpha_{k-1}, x_k^{(j)}, \alpha_{k+1}, \dots, \alpha_n\right); j = 1,2,\dots,N$. Here $\left[\tilde{y}_{\sim 1}^{(j)}, \tilde{y}_{\sim 2}^{(j)}, \dots, \tilde{y}_{\sim m}^{(j)}\right]^t; j = 1,2,\cdots,N$; are the realizations of the random vector $\widetilde{\boldsymbol{Y}}_{\sim i} = [\tilde{Y}_{\sim 1}, \tilde{Y}_{\sim 2}, \dots, \tilde{Y}_{\sim m}]^t$.
5) Estimate the jpdf of $\boldsymbol{Y}$ and $\widetilde{\boldsymbol{Y}}_i$,denoted by $\hat{p}_{\boldsymbol{Y}}(y_1, y_2, \cdots, y_m)$ and $\hat{p}_{\widetilde{\boldsymbol{Y}}_i}(\tilde{y}_1, \tilde{y}_2, \cdots, \tilde{y}_m)$ respectively, using multivariate kernel density estimator [28].
6) The total effect with respect to the input variable, X_i based on Hellinger distance can be estimated by evaluating the shift between the jpdfs of $\boldsymbol{Y}$ and $\widetilde{\boldsymbol{Y}}_i$, as calculated using the Eq. (3). Here, $\hat{p}_{\boldsymbol{Y}}\left(\tilde{y}_1{}^{(j)}, \tilde{y}_2{}^{(j)}, \cdots, \tilde{y}_m{}^{(j)}\right)$ and $\hat{p}_{\widetilde{\boldsymbol{Y}}_i}\left(y_1{}^{(j)}, y_2{}^{(j)}, \cdots, y_m{}^{(j)}\right)$ are evaluated using an appropriate interpolation method.

$$\hat{d}^2{}_H\left(\boldsymbol{Y}, \widetilde{\boldsymbol{Y}}_i\right) = 1 - 0.5\left\{\frac{1}{N}\sum_{j=1}^{N}\sqrt{\frac{\hat{p}_{\boldsymbol{Y}}\left(\tilde{y}_1{}^{(j)}, \tilde{y}_2{}^{(j)}, \cdots, \tilde{y}_m{}^{(j)}\right)}{\hat{p}_{\widetilde{\boldsymbol{Y}}_i}\left(\tilde{y}_1{}^{(j)}, \tilde{y}_2{}^{(j)}, \cdots, \tilde{y}_m{}^{(j)}\right)}} + \frac{1}{N}\sum_{j=1}^{N}\sqrt{\frac{\hat{p}_{\widetilde{\boldsymbol{Y}}_i}\left(y_1{}^{(j)}, y_2{}^{(j)}, \cdots, y_m{}^{(j)}\right)}{\hat{p}_{\boldsymbol{Y}}\left(y_1{}^{(j)}, y_2{}^{(j)}, \cdots, y_m{}^{(j)}\right)}}\right\} \quad (3)$$

7) The total effect with to respect to the input variable, X_i based on l_2 norm can be estimated using Eq. (4).

$$\hat{l}_2\left(\boldsymbol{Y}, \widetilde{\boldsymbol{Y}}_i\right) = \frac{1}{N}\sum_{k=1}^{m}\sum_{j=1}^{N}\left(Y_k^{(j)} - \widetilde{Y}_k^{(j)}\right)^2 \quad (4)$$

8) Repeat the steps (1) to (7) for $i = 1,2,\dots,n$.

In a similar manner, the main effect with respect to the input variable, X_i can also be evaluated by estimating the shift between the jpdfs of reference model, $\boldsymbol{Y}$ and altered model, $\widetilde{\boldsymbol{Y}}_{\sim i}$.

4 ILLUSTRATIONS

To exemplify the proposed sensitivity indices utilizing the Hellinger distance and l_2 norm, this study provides two examples. The first example examines a bivariate response function, while the second example investigates the deflection of a steel truss at two different nodes, which serves as the output function.

4.1 Example 1

Consider a bivariate output function represented by $\boldsymbol{Y} = f(X_1, X_2, X_3)$. Here $\boldsymbol{Y} = [Y_1, Y_2]^t$ is defined as $Y_1 = 2X_1 + 3X_2 + 3.5X_3$ and $Y_2 = 5 + 2X_1 + 7X_2X_3 + 3X_2^2 + 2X_1X_2X_3$. The uncertainties in the input variables, X_1, X_2, and X_3 are modelled as normally distributed random variables with the mean vector as $[3,4,5]^t$ and standard deviation vector as $[1,1,1]^t$. The coefficients of correlation between all the input variables are assumed to be 0.5.

The model distance-based global-local sensitivity analysis is performed for $\boldsymbol{Y}$ using distance measures such as Hellinger distance and l_2 norm for evaluating the total and main effects. The brute force MCS method is used to simulate the samples of X_1, X_2, and X_3. $1e^4$ samples are simulated for calculating the Hellinger distance-based sensitivity indices, whereas $1e^6$ samples are simulated for computing the l_2 norm based sensitivity indices. The sensitivity indices with respect to the input variables are tabulated in Table 2. The numbers in the parentheses indicate the rank of the input variables.

Variables	Based on Hellinger distance				Based on l_2 norm			
	Total effect		Main effect		Total effect		Main effect	
X_1	0.0314	(3)	0.8055	(3)	2.2191×10^3	(3)	17.263×10^3	(3)
X_2	0.4370	(1)	0.7243	(1)	9.4160×10^3	(1)	8.2169×10^3	(1)
X_3	0.3834	(2)	0.7466	(2)	3.5430×10^3	(2)	15.345×10^3	(2)

Table 2: Total and main effects evaluated for input random variables mentioned in Example 1.

From Table 2, X_2 is the most significant variable and X_1 is the least significant variable in both analysis. It can also be observed that the ranking of the input variables based on both Hellinger distance and l_2 norm follow the same order. It has to be noted that the total effects are ranked in descending order and main effects in the ascending order.

4.2 Example 2

Consider a simply supported steel truss as shown in Figure 2. The loads, denoted by $P_i; i = 1,2,\cdots,5$ are acting in vertical direction as shown in the figure. The cross-sectional areas of the inclined and horizontal members are denoted by A_1 and A_2 respectively whereas, the elastic moduli of the inclined and horizontal members are denoted by E_1 and E_2 respectively.

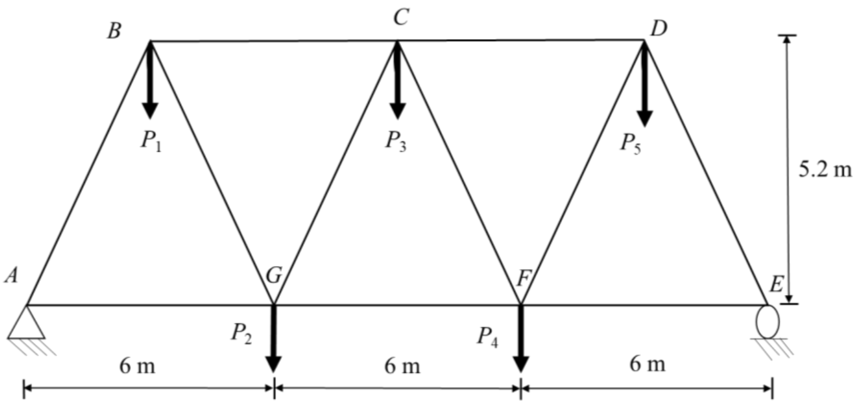

Figure 2: Truss model described in Example 2.

The multivariate sensitivity analysis is performed for two outputs - the vertical deflections at nodes, D and G, denoted as δ_D and δ_G respectively. These deflections are estimated using the Eqs. (5) and (6).

In this example, two cases are considered. In the first case, the elastic modulus and cross-sectional areas are treated as deterministic variables, keeping the loads as probabilistic variables. In the second case, all the variables are modelled as probabilistic variables.

$$\delta_D = \frac{1}{A_1E_1}(1.327\,P_1 + 2.658\,P_2 + 3.988\,P_3 + 5.318\,P_4 + 6.648\,P_5) + \frac{1}{A_2E_2}(1.95\,P_1 + 3.567\,P_2 + 4.504\,P_3 + 4.445\,P_4 + 3.056\,P_5) \tag{5}$$

$$\delta_G = \frac{1}{A_1E_1}(5.332\,P_1 + 10.665\,P_2 + 7.998\,P_3 + 5.332\,P_4 + 2.666\,P_5) + \frac{1}{A_2E_2}(4.438\,P_1 + 7.544\,P_2 + 7.986\,P_3 + 6.43\,P_4 + 3.546\,P_5) \tag{6}$$

4.2.1 CASE 1

The deterministic values considered for elastic modulus and cross-sectional area of inclined members are 210 GPa and 0.04 m^2 whereas, for horizontal members are 199 GPa and 0.09 m^2 respectively. The statistical properties of the uncertain load variables are detailed in Table 3. The loads $P_i; i = 1,2,\cdots,5$, are correlated with each other with a coefficient of correlation given by 0.5.

Notations	Variables	Distribution	Mean ($\times 10^3$ kN)	Standard deviation (kN)
X_1	P_1	Normal	23	2300
X_2	P_2	Normal	9.2	920
X_3	P_3	Normal	7.3	730
X_4	P_4	Normal	9.2	920
X_5	P_5	Normal	23	2300

Table 3: Statistical properties of the load parameters (Example 2).

The multivariate global-local sensitivity analysis is performed on the uncertain deflections. The total effects of the input variables based on Hellinger distance and l_2 norm are estimated using brute force MCS and these results are compared with exact solutions. The results are tabulated in Table 4. The numbers in the parentheses indicate the rank of the input variables.

Notations	Variables	Based on Hellinger distance				Based on l_2 norm			
		Exact		MCS		Exact ($\times 10^{-6}$)		MCS ($\times 10^{-6}$)	
X_1	P_1	0.0741	(2)	0.1904	(2)	4.546	(2)	4.544	(2)
X_2	P_2	0.0411	(3)	0.0653	(3)	2.668	(3)	2.667	(3)
X_3	P_3	0.0175	(5)	0.0112	(5)	1.335	(5)	1.335	(5)
X_4	P_4	0.0229	(4)	0.0119	(4)	1.507	(4)	1.507	(4)
X_5	P_5	0.6679	(1)	0.6791	(1)	6.377	(1)	6.377	(1)

Table 4: Total effect evaluated for load parameters (case 1, Example 2).

The analysis of Table 4 reveals that the input random variable P_5 is the most influential variable, while P_3 is the least significant variable in contributing to the uncertainties in the deflection at nodes D and G. Moreover, the results indicate that both distance measures yield similar outcomes when calculating the relative importance of the input variables.

4.2.2 CASE 2

In this case, all the input variables are modelled as probabilistic variables. The statistical properties of these variables are detailed in Tables 3 and 5.

Notations	Variables	Distribution	Mean	Standard deviation
X_6	A_1	Log- normal	0.04 m^2	0.004 m^2
X_7	A_2	Log- normal	0.09 m^2	0.009 m^2
X_8	E_1	Log- normal	210×10^6 kN/m^2	21×10^6 kN/m^2
X_9	E_2	Log- normal	199×10^6 kN/m^2	19×10^6 kN/m^2

Table 5: Statistical properties of the material properties (Example 2).

Table 6 presents the total and main effects with respect to the variables representing the load and material properties. It is observed the variable P_5 is found to be most influential variable and variable E_2 is the least significant towards the uncertainties contribution in the deflection at nodes, *D* and *G*. The ranking of the input parameters is found to be in similar order for both Hellinger distance and l_2 norm based distance measures.

No-ta-tions	Var-ia-bles	Based on Hellinger distance				Based on l_2 norm			
		Total effect		Main effect		Total effect ($\times 10^{-4}$)		Main effect	
X_1	P_1	0.2037	(2)	0.8226	(2)	4.774	(2)	0.0035	(2)
X_2	P_2	0.1661	(4)	0.8851	(4)	2.803	(4)	0.0038	(4)
X_3	P_3	0.0648	(6)	0.8894	(6)	1.407	(6)	0.0041	(6)
X_4	P_4	0.0939	(5)	0.8877	(5)	1.586	(5)	0.0040	(5)
X_5	P_5	0.2994	(1)	0.8169	(1)	6.663	(1)	0.0033	(1)
X_6	A_1	0.1852	(3)	0.8243	(3)	3.027	(3)	0.0037	(3)
X_7	A_2	0.0385	(8)	0.9417	(8)	0.748	(8)	0.0047	(8)
X_8	E_1	0.0623	(7)	0.9305	(7)	1.093	(7)	0.0046	(7)
X_9	E_2	0.0204	(9)	0.9577	(9)	0.099	(9)	0.0048	(9)

Table 6: Total and main effects evaluated for input variables (case 2, Example 2).

5 CONCLUSIONS

- This paper proposes the idea of model distance-based sensitivity analysis using statistical distance measures for quantifying uncertainties in response quantities of interest in engineering systems with multiple outputs.
- The proposed method introduces global-local response sensitivity indices that are computed using probabilistic distance measures based on the Hellinger metric and l_2 norm to identify significant input variables.
- The proposed method is able to deal with correlated and non-identically distributed random variables.
- The presented examples demonstrate the effectiveness of the proposed method in identifying significant input variables contributing to uncertainties in the response of the systems.

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