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CHANGE DETECTABILITY IN BAYESIAN SETTING

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Abstract. *This paper presents an approach to assess minimum detectable parameter changes based on Bayesian inference and the concept of highest posterior density interval. The method is developed for structural health monitoring problems, where observations of system outputs are used to infer knowledge about random system inputs. The analysis is based on linear Bayesian filters and the parameter change is defined as the shift in the mode of the distribution. For proof of concept, the framework is applied to a case study, that is, a numerical model of an offshore tower affected by marine growth. A functional Kalman filters is used to predict the minimum detectable changes, and for cross-validation, the prediction is validated based on a Markov Chain Monte Carlo simulation. The results show that, as long as the parameter can be satisfactorily identified, the minimum detectable parameter changes can also be accurately predicted. One of the advantages of the approach is that the minimum detectable parameter change can be predicted based on observations on the unchanged system. Moreover, it can be applied to a wide range of observed features, damage scenarios, and linear or non-linear systems. In contrast to existing approaches, the presented version is not restricted to Gaussian distributions.*

Keywords: Structural health monitoring, model updating, detectability, Bayesian inverse problem, Kalman filter, polynomial chaos expansion, uncertainty quantification

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1 INTRODUCTION

One of the purpose of Structural Health Monitoring (SHM) and Non-Destructive Test (NDT) applications is to detect a change in a quantity of interest which can not be directly observed. What can be directly observed are system outputs which are physically related to the quantity of interest, so that the observations can provide valuable knowledge about changes in the system inputs. However, uncertainties are involved, that affect not only the quantity of interest but also the observation. Since these uncertainties compromise the ability of the measurement system to detect anomalies, it is important to define the minimum detectable change that the measurement system is able to detect ideally before carrying out the experiment.

A literature review shows that a number of approaches have been developed to assess change or damage detectability, but most of them are empirical and limited to certain types of structures or damage scenarios [1, 2, 3]. Mendler et al. [4] developed an analytical method that can be universally applied to a wide range of structures and damage scenarios. The method is based on the concept of probability of detection (POD) and allows for the quantification of the minimum detectable damage. The approach was extended to damage localization [5], showing that there is a trade-off between optimizing the detectability and the localization resolution, and other factors. These methods are based on deterministic numerical models and do not consider uncertainties in the system parameters. In contrast, Marsili et al. [6] proposed an approach to evaluate the minimum detectable changes in a fully probabilistic framework that also allows for system parameters to be modelled as random variables. The approach is based on Bayesian inference and Kalman filters, which allow one to assess the detectability of changes before they occur. That means, that no data from the damage state is needed, and in addition, the method can also be applied to various damage scenarios and measurements, provided they are sufficiently sensitive to changes in the examined material parameters. The random variables are assumed to be normally distributed; however, this assumption may be not verified in the circumstances where the monitored parameter takes only positive values, as in the case of mass or mechanical parameters, or where the problem under consideration is nonlinear.

In this article, the framework developed in Marsili et al. [6] is extended to random variables with asymmetrical distributions. Linear Bayesian filters are used to quantify the uncertainty in the posterior distribution of the examined parameter before observations of system outputs are available. The highest density interval of the posterior distribution is then compared to a user-defined safety region, and a decision rule is introduced to clearly separate the damaged from the safe state. This separation is quantified through the change in the mode of the prior and the posterior, which is proposed as a measure for the minimum detectable parameter change. This approach is demonstrated based on a numerical case study on a offshore tower that is subject to marine growth. The polynomial chaos expansion based Kalman filter (PCE-KF) is used, which allows an analytic solution of the Bayesian inverse problem. A cross-validation is performed by applying the Markov Chain Monte Carlo (MCMC) algorithm. The paper is organized as follows: Section 2 presents the polynomial chaos expansion-based Kalman filters, and introduces the framework to quantify the minimum parameter change in a fully probabilistic setting. Section 3 contains an application to a numerical case study, and Section 4 draws some conclusions.

2 METHODS

2.1 The polynomial chaos expansion-based Kalman filter

The point of departure is a mechanical system which is modelled through a set of governing equations, i.e., partial differential equations. All decisive structural or material parameters are represented through a set of independent random variables $\mathbf{Q} \in \mathbb{R}^{N_p}$ with N_p the number of these parameters. In this paper, capital latin letters $\mathbf{Q}$ are used for random variables and small letters $\mathbf{q}$ are used for their realizations. It is assumed that a deterministic solver G exists (e.g., a finite element model) which transfers each input vector $\mathbf{Q}$ into a unique output vector $\mathbf{Y} = \mathrm{G}(\mathbf{Q}) \in \mathbb{R}^{N_f}$, where N_f is the number of observed system outputs. Since measurement errors are inevitable, the observable output $\mathbf{Z}$ is modelled as a linear combination of the observations and the error, i.e., $\mathbf{Z} = \mathrm{G}(\mathbf{Q}) + \mathbf{E}$. The distribution of the measurement error $\mathbf{E} \in \mathbb{R}^{N_f}$

$$\mathbf{E} = \mathbf{Z} - \mathrm{G}(\mathbf{Q}) \sim \mathcal{N}(\mathbf{0}, \mathbf{C_E}), \tag{1}$$

is assumed to have a zero mean vector and a diagonal covariance matrix $\mathbf{C_E} \in \mathbb{R}^{N_f \times N_f}$.
The Bayesian approach to the inverse problem seeks to update the random vector $\mathbf{Q}$ given a set of observations $\mathbf{z}$. This problem can be pursued by a variety of computational approaches. One particularly efficient method is to treat the Bayesian updating as a quadratic minimization problem, and to restrict the solution to linear measurable functions [7]. This leads to the linear Bayesian formula

$$\mathbf{Q}' = \mathbf{Q} + \mathbf{K}(\mathbf{z} - \mathbf{Y}), \tag{2}$$

where $\mathbf{Q}$ is the prior, $\mathbf{z}$ is the observation, and $\mathbf{Y}$ is the system outputs from a analytical model. The metric $\mathbf{K}$ is the Kalman gain [8]

$$\mathbf{K} = \mathbf{C_{QY}} \left[\mathbf{C_Y} + \mathbf{C_E}\right]^{-1}, \tag{3}$$

where $\mathbf{C_Y}$ is the system output covariance, $\mathbf{C_E}$ is the error covariance, and $\mathbf{C_{QY}}$ is the covariance between the system inputs and the system outputs. If the prior $\mathbf{Q}$ is Gaussian and the model G is linear, the posterior is Gaussian as well $\mathbf{Q}' \sim \mathcal{N}(\mu_{\mathbf{Q}'}, \mathbf{C}_{\mathbf{Q}'})$, and the posterior mean and the covariance can be computed according to the following formula [9]

$$\mu_{\mathbf{Q}'} = \mu_{\mathbf{Q}} + \mathbf{K}(\mathbf{z} - \mathrm{G}(\mu_{\mathbf{Q}})), \quad \mathbf{C}_{\mathbf{Q}'} = \mathbf{C_Q} - \mathbf{K}\mathbf{C}^{T}_{\mathbf{QY}}. \tag{4}$$

The random variables can be represented in different way, for example, through probability distribution functions (PDF), through samples, or through a functional approximation. In the latter, the system inputs, the system outputs, the measurement error and thus the observations are represented in a polynomial chaos expansion form

$$\hat{\mathbf{Q}} = \sum_{\mathcal{I}} \hat{\mathbf{q}}_\alpha \mathbf{H}_\alpha, \quad \hat{\mathbf{Y}} = \sum_{\mathcal{I}} \hat{\mathbf{y}}_\alpha \mathbf{H}_\alpha, \quad \hat{\mathbf{Z}} = \sum_{\mathcal{I}} \hat{\mathbf{z}}_\alpha \mathbf{H}_\alpha, \quad \hat{\mathbf{E}} = \sum_{\mathcal{I}} \hat{\mathbf{e}}_\alpha \mathbf{H}_\alpha, \tag{5}$$

where $\hat{\mathbf{q}}_\alpha$, $\hat{\mathbf{y}}_\alpha$, $\hat{\mathbf{z}}_\alpha$, $\hat{\mathbf{e}}_\alpha$ are coefficients and $\hat{\mathbf{H}}_\alpha$ represents the generalised multi-variate orthogonal polynomials. The letter $\mathcal{I}$ represents the set $\boldsymbol{\alpha}$ of multi-indices such that the following expression holds [10]

$$\mathcal{I} := \boldsymbol{\alpha} = (\alpha_1, ..., \alpha_j, ...) \;\;\Big|\;\; \alpha_j \in \mathbb{N}_0^{(\mathbb{N})}, \quad |\boldsymbol{\alpha}| = \sum_{j=1}^{\infty} \alpha_j < \infty. \tag{6}$$

A finite subset of $\mathcal{I}$ is considered for computational reasons, which implies that the expansion is truncated after the polynomial of order p. In this case, the linear Bayesian formula of Eq. (2) can be rewritten in the following form [11, 12]

$$\hat{\mathbf{Q}}' = \hat{\mathbf{Q}} + \mathbf{K}(\hat{\mathbf{z}} - \hat{\mathbf{Y}}), \tag{7}$$

where $\mathbf{K}$ is the Kalman gain from Eq. (3) evaluated analytically with the following covariances

$$\mathbf{C_Y} = \sum_{\alpha>0} \alpha!\hat{\mathbf{y}}_\alpha(\hat{\mathbf{y}}_\alpha)^\mathrm{T}, \quad \mathbf{C_E} = \sum_{\alpha>0} \alpha!\hat{\mathbf{e}}_\alpha(\hat{\mathbf{e}}_\alpha)^\mathrm{T}, \quad \mathbf{C_{QY}} = \sum_{\alpha>0} \alpha!\hat{\mathbf{q}}_\alpha(\hat{\mathbf{y}}_\alpha)^\mathrm{T}. \tag{8}$$

Only the posterior mean value $\boldsymbol{\mu}_{\mathbf{Q}'}$ depends on the value of the observation, while the posterior covariance $\mathbf{C}'_\mathbf{Q}$ is independent of it. This property makes it possible to predict the posterior covariance $\mathbf{C}'_\mathbf{Q}$ analytically, and before any data is available, on the basis of the prior covariance $\mathbf{C_Q}$, the measurement error $\mathbf{C_E}$ and covariance $\mathbf{C_{QY}}$ between the system input $\mathbf{Q}$ and the observable system output $\mathbf{Y}$.

2.2 Change detectability in a probabilistic setting

This section develops a framework to determine the minimum detectable parameter change. For the sake of simplicity, the following explanation focuses on uni-variate problems which are characterized by a single parameter Q to be updated. Since the monitored parameters are stochastic before and after updating, it is non-trivial to define one value as the minimum detectable parameter changes. For Gaussian distributions, the change in the mean value could be defined as the parameter change [6]. In case of asymmetrical distribution, the change in the mode might be a more suitable metric, as the mode describes the most likely parameter value. Therefore, changes in a single examined parameter ΔQ are modelled through

$$\Delta Q = m_{Q'} - m_Q, \tag{9}$$

where m_Q is the mode of the prior random variable Q and $m_{Q'}$ of the posterior.
The problem of detecting damage or change is often approached through the statistical act of testing a hypothesis. In the Bayesian framework, the probability of a hypothesis is inferred from the evidence, that is, the available data. The prior belief is the prior probability of the hypothesis, while the posterior belief is the test statistic that allows one to accept or reject the hypothesis. To reject a hypothesis, for example to issue a damage diagnosis, three additional elements must be defined: a safety interval, a decision criterion and a decision rule.

- Definition of the safety interval: An interval is constructed around a parameter value of interest ω_0, which consists of a set of values q equivalent to ω_0 for the practical purpose of the applied real-world decision [13]. The parameter value ω_0 is a reminiscence of the null value in the frequentist approach to hypothesis testing; in this application, it corresponds to the mode m_Q of the prior distribution. In damage and change detectability, the region of practical equivalence represents the safety interval, and its thresholds Q^-_{thres} and Q^+_{thres} can be referred as lower and upper safety threshold, respectively.

- Definition of the decision criterion: To enable an automated change or damage detection, a decision criterion is defined, representing a synthesis of the posterior distribution. In this paper, the highest density interval of the posterior distribution (highest posterior density interval, HPDI) is employed as a decision criterion. For a given confidence level $\alpha \in (0, 1)$, the $100(1 - \alpha)\%$ HPDI is the shortest interval which contains a proportion $1 - \alpha$ of the probability mass of the posterior distribution.

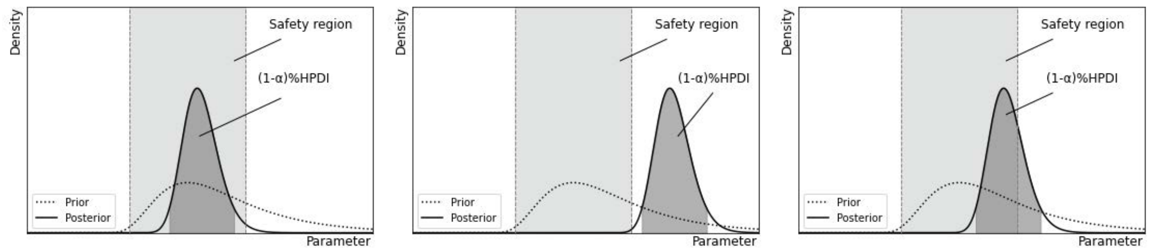


Figure 1: Defining (i) the safe state, (ii) the damaged state, and (iii) the ambivalent state

- Definition of the decision rule: The decision rule stems from comparing the safety interval and the HPDI. The following three cases can be distinguished, see Fig. 1, leading to three different decisions:
  (i) If the HPDI is within the safety interval, the parameter is in the safe state. The likelihood of parameter changes is low, and no alarm should be issued by the monitoring system.
  (ii) If the HPDI is outside the safety interval, the parameter is in damaged state. The likelihood that the parameter has changed is high, and the automated monitored system should issue an alert.
  (iii) If the HPDI and the safety interval overlap, no clear decision about the state of the parameter can be made. The parameter is in an ambivalent state.

A limit state is reached when the lower boundary of the HPDI matches with the upper safety threshold. This is the minimum detectable parameter *increase* ΔQ^{+}_{min} that can be detected based on the selected confidence level α. Equivalently, the lower safety threshold and the upper boundary of the HPDI define the minimum detectable parameter *decrease* ΔQ^{-}_{min}.
Two additional limit situations also occur in the case where the HPDI is contained within the safety interval, with the lower threshold matching the lower threshold of the safety interval, or vice versa, with the upper threshold matching the upper threshold of the safety interval. In these cases, the change can be defined as the maximum acceptable parameter fluctuation.
The width of the HPDI depends on the established confidence level, that is, the probability for which the parameter value falls within the interval. Consequently, for each predefined confidence level, a HPDI can be defined. By defining the confidence level, and predicting the posterior covariance by applying the formula 4, it is then possible to predict the width of the HPDI before any output of the system is observed. This fact, integrated into the framework defined in Section 2.2, makes it possible to determine the required change in the parameter to make a decision about its state, that is, to define the distance of the posterior mode $m_{Q'}$ from the safety threshold for which the parameter can be declared in a state of safety or in a state of damage.

3 APPLICATION

3.1 Detectability of mass changes in a tower subject to marine growth

The framework developed in Section 2.2 is applied to a case study, namely a jacket-type tower affected by marine growth (Fig. 2), that is, a mass increase at submerged structural components due to shellfish, algae, etc. This case study has already been considered in Marsili et al. [6]. Since mass is a variable that cannot assume negative values, it may be more appropriate

to use a probability density function defined on exclusively positive values, such as a lognormal distribution with mean of 1000 t and a standard deviation of 200 t, $Q \sim \mathcal{LN}(13.7959, 0.0392)$, see Fig. 3. The purpose of this application is to demonstrate that by defining the value of the confidence level α and applying Eq. 4, it is possible to assess the width of the $100(1-\alpha)\%$ HPDI before any observations of system outputs are available. By imposing the boundary condition in which the extremes of the safety interval and HPDI match, it is possible to determine the parameter change necessary to make a clear decision about the state of the parameter. In other words, this allows us to determine what is the distance of the mode from the safety threshold such that a decision can be made on the state of the parameter (Fig. 4).
The measurements provided by five inclinometers located as showed in Fig. 2 are used directly as damage-sensitive features. As it has been already clarified in Marsili et al. [6], the mass does not change but for the computation of displacements and inclinations, the mass is transformed into an equivalent force, which may be counteracted by changing tides and varying water buoyancy forces. In the reference configuration, the mean inclination is

$$\mu_Y = [-0.0332, 0.0332, 0.02594, -0.00107, 0.00107] \tag{10}$$

millidegrees and the standard deviation of the measurement noise is 10% the mean inclination. The coefficients of the polynomial expansion can be estimated according to different methods, e.g., interpolation, regression, projection [14]. The latter is used in this paper, by considering a polynomial order of $p = 5$.

3.2 Validation of the proposed approach

In this section, it is shown that the framework developed in this paper allows accurate prediction of the mode of the posterior. A confidence level $\alpha = 0.05$ is assumed to define the HPDI and consequently to make the decision about the state of the parameter. This implies that the value of the random variable has a 95% probability of falling into the HPDI.
A distinction is made between a critical parameter, where the parameter is connected to a ultimate limit state and a change in its value may lead to important negative consequences, and a non-critical parameter, if on the contrary the parameter is connected to a serviceability limit state and a change in its value has less significant effects. In the former case, and with the intention of reducing the risk associated with a change in the parameter, it is of interest to assess what is the maximum parameter fluctuation for which the state of the parameter can still be called safe. Conversely, in the second case, the focus is on assessing the minimum change necessary to issue a damage diagnosis. In this application, both limiting cases are investigated for which a decision on damage state and safe state can be issued.
The next step is to define the safety interval, that is, the set of parameter values that are equivalent to the value of interest ω_0 for the practical purpose pursued by this application. The value of interest is considered to be the mode of the prior distribution m_Q, and the region of safety is constructed symmetrically around this value. The definition of the width of the safety interval depends on the width of the posterior distribution: if this region is too narrow with respect to the width of the a posterior distribution, it will never be possible to issue a safe diagnosis. In a first application, the width of the safety region is established to be 3 times the 95%HPDI (Case 1). Being $95\%\mathrm{HPDI}^+$ and $95\%\mathrm{HPDI}^-$ the upper and lower extreme of the 95%HPDI respectively, this implies that $Q^+_{\mathrm{thres}} - Q^-_{\mathrm{thres}} = 3 \times (95\%\mathrm{HPDI}^+ - 95\%\mathrm{HPDI}^-)$. Fig. 5a and 5b show the result of the validation for the case where the parameter is in a safe state and is in a damaged state, respectively.

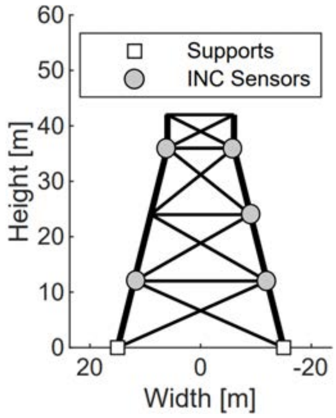


Figure 2: Jacket-type tower modeled in 2-D and location of the five inclinometers

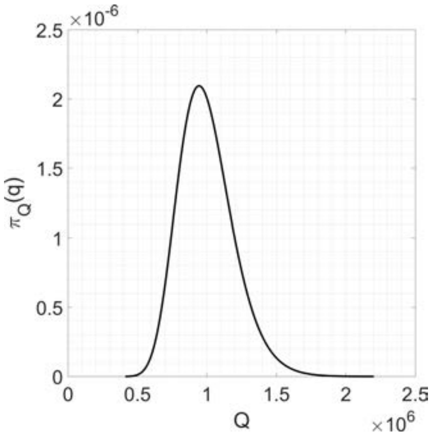


Figure 3: Prior PDF for the random variable ”mass”

	Type	Material	Cross section			
		E (MN/m^2)	A (m^2)	I (m^4)	EA (MN)	EI (MNm^2)
Pillars	Beam	210000	1.0	0.146	210000	307000
Diagonals	Truss	210000	0.5	0.147	105000	307000

Table 1: Properties of the tower model.

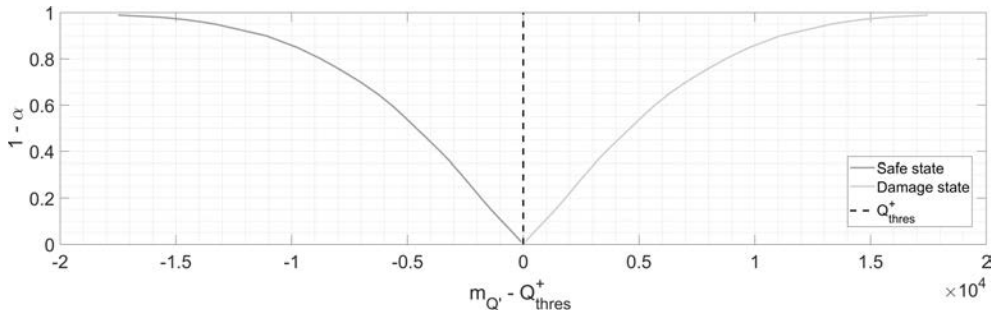
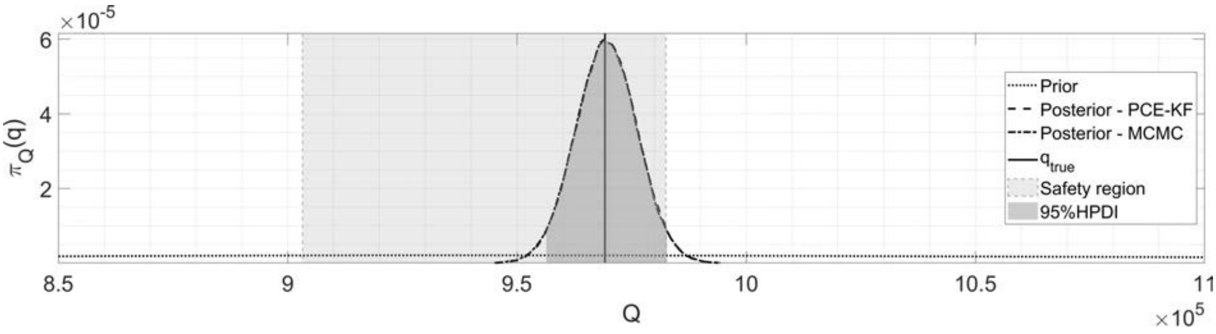


Figure 4: Distance of the posterior mode $m_{Q'}$ from the safety threshold Q^+_{thres} as a function of $1-\alpha$

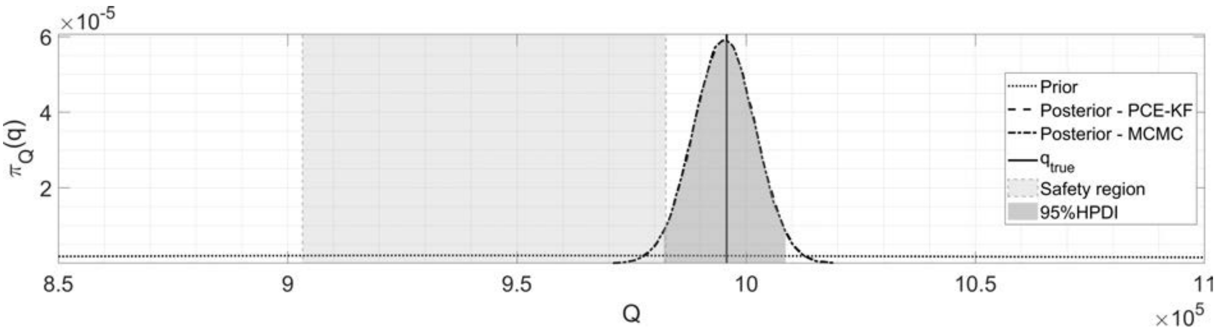
	Safety region		Safe state			Damage state		
	ω_0	Q^+_{thresh}	q_{true}	$m_{Q'}$	$95\%HPDI^+$	q_{true}	$m_{Q'}$	$95\%HPDI^-$
Case 1	0.9429	0.9825	0.9692	0.9693	0.9828	0.9692	0.9705	0.9828
Case 2	0.9429	1.0221	1.0091	1.0080	1.0212	1.0354	1.0334	1.0205

Table 2: Results of the validation (all quantities are given in Mt)

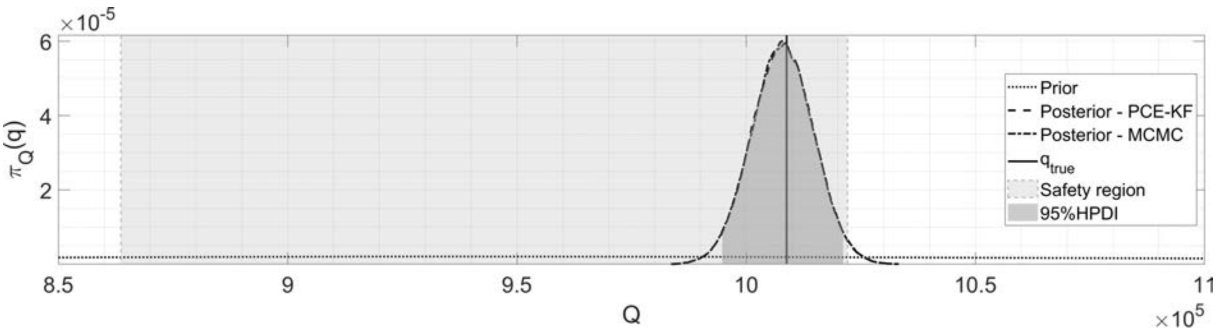
As we can see from Fig. 5a and 5b, the 95%HPDI is positioned completely inside and outside the safety region, with the extreme of the interval coinciding with very good approximation with the safety threshold. The true value of the parameter q_{true} also corresponds with good approximation to the mode of the posterior distribution. These results are also reported in Table



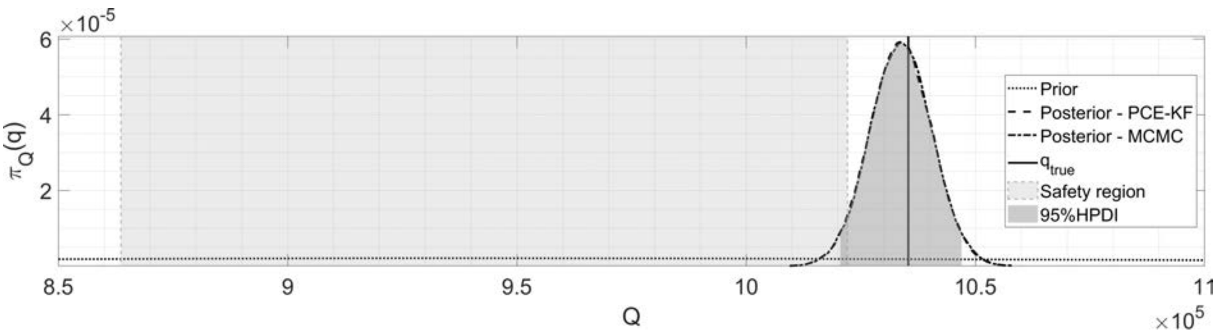
(a) Case 1: Safe state



(b) Case 1: Damaged state



(c) Case 2: Safe state



(d) Case 2: Damaged state

Figure 5: Validation of the proposed approach

2. As the problem dimension is small, it is possible to perform cross-validation by applying the MCMC algorithm, and to reduce the computation time, the proxy model can be used instead of the forward model: the result is in total agreement with that obtained by applying PCE-KF, highlighting the fact that the posterior covariance is correctly identified by the PCE-KF method. The validation is also performed considering a larger width of the safety region, corresponding to six times 95%HPDI (Case 2, $Q^{+}_{\mathrm{thres}} - Q^{-}_{\mathrm{thres}} = 6 \times (95\%\mathrm{HPDI}^{+} - 95\%\mathrm{HPDI}^{-})$). The results showed in Fig. 5c and 5d reveal that the prediction is to be satisfactory if the parameter is in a safe state; however, in the case of damage diagnosis (Fig. 5d), the inaccuracy in parameter identification is greater, as demonstrated by a greater discrepancy between the posterior mode and the true value of the parameter. This leads to less satisfactory results in the prediction of the change, as also a greater discrepancy exists between the Q^{+}_{thres} and $95\%\mathrm{HPDI}^{-}$, resulting into an overlapping between the safety region and the 95%HPDI.

According to these results, it is possible to conclude that the approach illustrated in this paper successfully predicts parameter change if parameter identification is also possible with satisfying accuracy. However, due to the limitations of the method, which have been clarified in Landi et al. [15], and the difficulties related to system and parameter identification in inverse problem, this is not always possible: in these cases, it is necessary to expect that the prediction of the detectable change will also be inaccurate. This fact could lead to unnecessary additional data collection or even to an incorrect decision about the state of the parameter. Consequently, it is recommended to test the method's ability to identify the parameter in the range of values significant for the application under consideration, and possibly to improve the identification of the parameters before using this approach to predict detectable changes.

4 CONCLUSION

This paper proposes a procedure to evaluate detectable parameter changes in probabilistic setting, also considering random variables having asymmetric distribution. The method is based on Bayesian inference, and it is suitable for any problem in which system outputs are used to infer knowledge about random system inputs. The Bayesian updating is performed applying the linear Bayesian filter: this allows to evaluate the posterior covariance before observations of system outputs are available, and thus to predict the width of the highest posterior density interval. This results into a prediction of the required distance of the posterior mode from a safety threshold according to which a decision about the state of the random parameter can be taken.

An application of the proposed procedure to a numerical case study, a tower affected by marine growth, is developed. In the application, both cases of critical parameter and non-critical parameter were treated, also considering different widths of the safety interval. According to the obtained results, the detectable change can be accurately predicted in most of the cases. However, if a discrepancy between the posterior mode and the true value of the parameter exists, this results in a discrepancy between the threshold of the highest posterior density interval and the safety threshold. Therefore, the detectable change can be successfully predicted if the structural identification based on the linear Bayesian filter is sufficiently accurate, resulting in negligible discrepancy. If this discrepancy is not acceptable, the structural identification process needs to be improved, before the method proposed in this article can be applied successfully.

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